Entrepreneurship, Knowledge, and the Industrial Revolution

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Abstract

Mustafa Aykut Attar: Entrepreneurship, Knowledge, and the Industrial Revolution
(Under the direction of Lutz A. Hendricks)

This dissertation constructs and studies a simple unified growth model that explains the timing of the industrial revolution through entrepreneurship and entrepreneurs' previously unexplored role in the accumulation of useful knowledge.

Three premises are responsible for the main results: First, inventions and discoveries, i.e. two forms of useful knowledge, are differentiated such that a larger stock of discoveries implies a higher level of inventive productivity. Second, entrepreneurs own and manage the firms operating in the innovative sector of the economy, and they thus may find it optimal to spend some of their scarce time endowment to inventive activity by decreasing the time allocated to routine management otherwise. Third, the stock of useful discoveries expands in time through the process of collective discovery. Entrepreneurs, during their lifetime, serendipitously perceive new useful discoveries and share what they discover with each other in their common social environment.

Two key results are that (i) the optimal level of inventive effort by entrepreneurs is zero if the stock of useful discoveries is sufficiently small, and (ii) an industrial revolution, i.e. an endogenous switch from zero to positive inventive effort, is an inevitable outcome of the process of collective discovery even though it might be delayed for long epochs of stagnation. Population growth and structural transformation, i.e. two well-documented aspects of the transition from stagnation to growth, are not only affected by technological progress as usual but also determine how fast the economy moves towards its invention threshold.

Calibrated to match some key data moments of England's economic development during the last 350 years, the model performs reasonably well in explaining the main patterns of the transition from stagnation to growth, i.e. the demographic transition, urbanization, industrialization and the acceleration of technological progress. Counterfactual experiments show that even small deviations from the benchmark model may create large effects on the timing of the industrial revolution.

To all my teachers

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Whether we define the entrepreneur as an "innovator" or in any other way, there remains the task to see how the chosen definition works out in practice as applied to historical materials. In fact it might be argued that the historical investigation holds logical priority and that our definitions of entrepreneur, entrepreneurial function, enterprise, and so on can only grow out of it *a posteriori*. Personally, I believe that there is an incessant give and take between historical and theoretical analysis and that, though for the investigation of individual questions it may be necessary to sail for a time on one tack only, yet on principle the two should never lose sight of each other. In consequence we might formulate our task as an attempt to write a comprehensive history of entrepreneurship.

Joseph A. Schumpeter

From his 1949 address at the Research Center in Entrepreneurial History
Cited by Lazonick (1991, pp. 271-272)

Chapter 1

Introduction

Why did the Industrial Revolution start when it did? Why not earlier or later? Why was the stagnation of living standards so prolonged? Which factors did keep today's developed societies and others in a quasi-trap of poverty for several millennia?

In the last decade, many growth theorists have returned to this timing question with the methodology of Galor and Weil's (2000) *Unified Growth Theory*: A unified model is the one that not only features stagnation and growth equilibria but also accounts for the factors that trigger and govern the *gradual* transition from the former to the latter.

This dissertation constructs and studies a simple unified growth model that explains the *timing* of the Industrial Revolution through *entrepreneurship* and entrepreneurs' role for the accumulation of *useful knowledge*. The growth of living standards in the model is due to new inventions created by entrepreneurs who behave very much like Schumpeter's (1934) "entrepreneur-inventor"s, and the start of the industrial revolution is an endogenously occurring switch from an equilibrium regime of zero inventive effort to that of positive inventive effort.

What motivates the emphasis on entrepreneurial invention, in addition to the Schumpeterian view of the first Industrial Revolution suggested by Solo (1951), Baumol (1990), Murphy et al. (1991) and Mokyr (2010), is Meisenzahl and Mokyr's (forthcoming) prosopographical evidence on 759 British inventors born between 1660 and 1830. Among 598 inventors with a known business ownership status, only 88 inventors (around 15%) were employed as non-managers, and 467 of them (around 78%) were *business owners*. The latter statistic suggests that understanding the role of inventors who were incentivized by profit motive during the Industrial Revolution may be of prime importance, and this dissertation develops a simple model that makes entrepreneurs' role explicit unlike existing unified growth models.

Three premises are responsible for the main results: First, building on Mokyr's (2002) theory of useful knowledge, *inventions* and *discoveries* are differentiated such that, for a given level of effort directed to inventive activity, it is less likely to be successful in achieving a given number of inventions if the number of available discoveries is smaller. By this premise, recently exploited by O'Rourke et al. (2008) and Strulik (2009) as well, the model endogenizes the productivity term of a standard invention technology that exhibits constant-returns-to-scale with respect to its rival labor input. Second, entrepreneurs in the model establish and manage their firms in a perfectly competitive sector that produces the single consumption good in the economy. That entrepreneurs appropriate positive profit by managing their own firms implies that it may be optimal for entrepreneurs to allocate some of their scarce time endowment to inventive activity while decreasing the time spent on routine management, hence the term *entrepreneurial invention* as in Grossmann (2009). Finally, the stock of useful discoveries expands in time through the process of *collective discovery*. Entrepreneurs, during their lifetime, serendipitously perceive new useful discoveries, i.e. new knowledge components about natural phenomena underlying the production processes but not being themselves inventions, and share what they discover with each other in their common social environment. This premise is a way to formalize, albeit imperfectly, what Mokyr (2002) calls *industrial enlightenment*, and it is motivated, among others, by Jacob (1997), Bekar and Lipsey (2004) and Landes (2006) who emphasize the creation and the diffusion of useful knowledge among British/European entrepreneurs and capitalists.

Two key results follow from these three premises. First, there exists an invention threshold: If the stock of useful discoveries is sufficiently small, given (exogenous) longevity, the optimal level of inventive effort by entrepreneurs is zero. Second, the endogenous mass of entrepreneurs through collective discovery determines how fast the economy moves to its invention threshold. Provided that an industrial revolution is *possible*, it is an *inevitable* outcome of the process of collective discovery even though it might be delayed for long epochs of stagnation. The question is thus what factors explain the mass of entrepreneurs.

To answer this in a parsimonious way within a unified growth framework, the model incorporates two well-documented aspects of the transition from stagnation to growth, i.e. *population growth* and *structural transformation*, since these determine the mass of entrepreneurs through the use of economy-wide resources for different tasks.

The model closely follows Jones (2001), Strulik and Weisdorf (2008) and de la Croix and Licandro (2009) to deliver the following story of demographic change in the simplest way: Population growth is endogenous through optimal fertility choice in a two-generation framework. Given exogenous patterns of child survival and adult longevity, the endogenous evolution of economic prosperity governs the dynamics of a demographic transition. Specifically, a minimum consumption constraint implies an upper limit of fertility when the economy is sufficiently poor. An increase in incomes, within this regime of fertility choice, leads to higher fertility. Once the economy becomes sufficiently rich, however, this income effect vanishes completely. The optimal level of fertility declines with technological progress since time, becoming more expensive, increases the unit time cost of a born child. Parents' strict preference towards reproductive success, i.e. leaving at least one surviving child, implies a constant *level* of population for sufficiently high income levels. Since earth as a closed system has a finite carrying capacity of people in reality, a constant level of population in the long-run is a desired feature. Yet, contrary to Peretto and Valente's (2011) model that explicitly considers the role of natural resource scarcity, the model simply predicts a constant level of population in the long-run through preferences towards reproductive success. The model also deviates from the quality-quantity trade-off models of Galor and Weil (2000) and Lucas (2002) by non-homothetic preferences that eliminate the income effect on fertility completely for sufficiently high income levels. This is, of course, not to say that the endogenous natural resource scarcity and the quality-quantity trade-off are not important for the demographic transition, but the model is greatly simplified by these deviations and still returns an empirically plausible story of demographic change.

The way in which the model incorporates structural transformation is similar to that of Hansen and Prescott (2002) and Bar and Leukhina (2010a) such that (i) a single consumption good is produced with two technologies, i.e. traditional/rural and modern/urban technologies, and (ii) the traditional technology admits land as an input in fixed supply. In contrast with Hansen and Prescott (2002) and Bar and Leukhina (2010a), on the other hand, (i) technological progress in the modern sector is endogenous through entrepreneurial invention and (ii) a spillover effect from the modern to the traditional sector endogenizes productivity change in the traditional sector as in Desmet and Parente (2009) at some period after the industrial revolution. Under an inequality restriction that governs the growth of relative productivity in the long-run, the traditional sector faces an eventual demise. The mapping from the model to the data associates the declines of the labor and the output shares of the traditional sector in the model respectively with the rises of urbanization and industrialization as in Bar and Leukhina (2010a).

The trajectory of economic development this dissertation predicts, consistent with the conventional wisdom of unified growth theory, is as follows: Adult longevity, child survival probability, the stock of useful discoveries, and urbanization rate are all at their historically minimum levels in the very beginning. Inventive activity is not optimal, and the modern/urban sector is small. Poverty is severe, limiting net fertility and population growth. In time, however, the minuscule increases in adult longevity and child survival probability, if any, minimally increase net fertility, and the stock of discoveries keeps expanding albeit at a still slow rate due to the small number of entrepreneurs. This process continues for sufficiently many periods to eventually make inventive effort optimal. An industrial revolution starts. Increasing wages imply faster urbanization and higher gross/net fertility first. Once productivity is sufficiently high, however, adult individuals find it too costly to produce too many children and limit fertility. Urbanization on the other hand keeps accelerating to make the mass of entrepreneurs even larger and collective discovery even more productive. Thus, industrialization accelerates. The asymptotic equilibrium features perpetual growth of labor productivity and output per capita with a strictly positive asymptotic growth rate and a constant level of population.

The quantitative analysis of the model calibrates the structural parameters and the initial values of state variables to match the observed patterns of economic development in England from 1650 to 2000. England is not only the first industrialized economy but also the one for which a sufficiently large set of data exists. The simulations for the period from 1650 to 2000 show that the model economy performs reasonably well in explaining population and output per capita dynamics and the rises of urbanization and industrialization. The calibrated model is next used for some counter-factual experiments on the timing of the industrial revolution as in Desmet and Parente (2009). The results, interpreted only as suggestive and only for the timing of England's industrial revolution, indicate that small deviations from the benchmark model may create large timing effects.

The contribution of this dissertation to the literature can be described as follows: First and the foremost, the dissertation shows that thinking a bit seriously about the productivity term of an otherwise standard invention technology and bringing the entrepreneur back to the scene of economic development allow us to understand why purposeful invention may not be optimal for a very long episode of history and why an industrial revolution is inevitable. Entrepreneurs' dual role for inventions and discoveries in a unified growth framework, motivated strongly by the anecdotal and prosopographical evidence, is the previously unexplored mechanism proposed in this dissertation. Second, the model eliminates both *the weak* and *the strong scale effects* of population size as in Grossmann (2009). The absence of the weak scale effect is important because the role of population size during the modern growth regime remains ambiguous in most unified growth models. The strong scale effect, on the other hand, is ruled out as the mass of entrepreneurs plays the role of the horizontal dimension of technological progress of the second-generation Schumpeterian models of Young (1998), Peretto (1998a), Aghion and Howitt (1998, Ch. 12) and Dinopoulos and Thompson (1998). Third, due to stochastic invention and ex post heterogeneity across firms in the modern sector, the model explains, unlike other unified growth models, (i) why more innovative firms on average are larger and why the size distribution of innovative firms are skewed; two well-known regularities most recently reiterated, respectively, by Akcigit and Kerr (2010) and Klette and Kortum (2004). Finally, although the model does not incorporate human capital in its usual sense, that it does not crucially require a rise in the demand for embodied skills for the industrial revolution to start, as that of O'Rourke et al. (2008), answers Clark's (2005) criticism of unchanged skill premium during the first Industrial Revolution.

The plan of the dissertation is as follows: The next chapter provides a discussion of the related literature to more clearly locate the contribution of this dissertation. Chapter 3 introduces the model economy. Chapter 4 defines and analyzes the uniquely existing static, dynamic and asymptotic equilibria of the model and then characterizes a possible equilibrium path from a unified growth perspective. Chapter 5 provides some tentative (analytical) results on the timing of the industrial revolution. Chapter 6 presents a quantitative analysis of the model economy. Chapter 7 provides a discussion of some aspects of the model and manipulates over some possible extensions. Chapter 8 concludes with some final remarks.

Chapter 2

The Related Literature

This chapter presents a discussion of the related literature in two sections: First, the notable contributions to the literature studying the transition from stagnation to growth are mentioned, and the relation of the dissertation with this literature is specified. Second, scholars who have contributed earlier on the three fundamental premises of the model, i.e. the discovery-invention distinction, entrepreneurial invention, and collective discovery, are credited, and the contribution of this dissertation is described.

2.1 From Stagnation to Growth

Poverty trap models of Murphy et al. (1989), Azariadis and Drazen (1990), Becker et al. (1990) and Matsuyama (1991) predict multiple steady-state equilibria, but these models leave the mechanism that explains the transition from stagnation to growth unexplored.[1] An early study that deals explicitly with the transition is of Goodfriend and McDermott (1995) who argue that there exists a critical mass of population that makes industrialization efficient through the market size effect. Tamura (1996) extends the model of Becker et al. (1990) by a clearer account of transitional dynamics. Building on

1. Each of these models focuses on a different mechanism that generates development thresholds and multiple equilibria. Murphy et al. (1989) emphasize the role of demand spillovers across industries by formalizing the Big Push argument. Azariadis and Drazen (1990) study the externalities in the accumulation of human capital that create threshold effects. Building on the fertility theory of Becker and Barro (1988), Becker et al. (1990) utilize the quality-quantity trade-off of human capital accumulation to generate Malthusian stagnation and modern growth as two (stable) steady-state equilibria. Matsuyama (1991) develops a two-sector model of occupational choice with increasing returns in manufacturing.

Azariadis and Drazen's (1990) model, Arifovic et al. (1997) endow agents by adaptive learning that eventually triggers the takeoff. Acemoglu and Zilibotti (1997) explain the minuscule rates of capital accumulation and economic growth during the early stages of development by the absence of complete financial markets that diversify risk. All these early models, however, fail to account for the demographic transition.

Galor and Weil's (2000) model, for the first time, successfully explains not only the prolonged Malthusian stagnation and the endogenous occurrence of an industrial revolution but also the demographic transition. The authors show that, once the Malthusian constraint of subsistence consumption and the quality-quantity trade-off of human capital accumulation in the fashion of, e.g., Becker et al. (1990) are conveniently integrated with the skill-biased technological change and with the slow pace of technology-population interaction as in Boserup (1965) and in Kremer (1993), the desired result follows; a gradual and endogenously occurring phase transition from stagnation to growth that does not require a major shock.

Notable contributions to the literature include that of Hansen and Prescott (2002) who study the decline of the land-based sector and the rise of industrialization within a neoclassical framework with exogenous technological progress. Lucas (2002) merges this framework with endogenous fertility choice and human capital accumulation. The emphasis on the demographic transition as a result of endogenous fertility choice is maintained by Jones (2001) who extends the analysis, for the first time, by a very simple formulation of incentives to innovate. The decline of agriculture is studied (i) within the framework of Hansen and Prescott (2002) but with endogenous growth in manufacturing sector by Kögel and Prskawetz (2001) and (ii) within Galor and Weil's (2000) framework by Tamura (2002). The analysis of the demographic transition is extended further by Hazan and Berdugo (2002) and Doepke (2004) who introduce child labor, by Boldrin and Jones (2002) who revisit the old-age security hypothesis with game theoretical notions, and by Lagerlöf (2003) who endogenizes mortality. Galor and Moav (2002) present an evolutionary unified model of economic growth in which the distribution of the valuations for human capital accumulation across individuals is subject to natural selection.[2]

A majority of unified growth models either do not utilize the *invention* of new products and new processes as the source of technological progress at all or productivity change is incorporated in a reduced-form way such that the mechanism of endogenous technological progress remains as a black-box. Motivated partly by this dissatisfaction and partly by the critiques of some economic historians such as Mokyr (2002), Crafts (2005), Lipsey et al. (2005) and Clark (2007) who keep stressing the role of inventions for understanding the transition from stagnation to growth, some recent unified growth models explicitly deal with the incentives leading agents to innovate, be them individual "consumer/producer"s or firms. O'Rourke et al. (2008) study the transition from unskilled-labor-biased technological change to the skill-biased technological change. Desmet and Parente (2009) put the emphasis on the evolution of market competition that, once sufficiently intensified, activates purposeful innovation by monopolistic firms. Milionis and Klasing's (2009) model, featuring "consumer/producer"s, explains why human capital accumulation eventually leads to purposeful invention. In Bar and Leukhina (2010b), individuals optimally choose the time they spend on innovative activities while the main emphasis is directed to the effects of mortality on knowledge transmission. The last but not the least, Madsen et al. (2010) show that the last four hundred years of British economic growth can best be understood as a story of endogenous technological progress.

The model constructed in this dissertation contributes to this second strand of the unified growth literature that endogenizes technological progress with an explicit invention technology and profit-seeking agents. The model is truly unified in the sense that the gradual phase transition from stagnation to growth occurs endogenously. Most importantly, the new mechanism proposed features entrepreneurs' dual role for inventions and discoveries.

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2. Unified growth literature is by now large and diversified, and a complete review is not central for the purposes of this dissertation. See Galor (2005, 2010) for two useful surveys of the literature.

Table 2.1: Two Forms of Useful Knowledge in Mokyr's (2002) Theory

Propositional Knowledge	Prescriptive Knowledge
Epistemic	Technical
Laws and Principles	Blueprints and Recipes
What-Knowledge	How-Knowledge
Discoveries	Inventions

2.2 Three Fundamental Premises

The conceptual framework of useful knowledge that this dissertation builds upon, with discoveries and inventions being distinct knowledge forms, is due to Mokyr (2002). In his theory, discoveries are propositional forms of knowledge that do not have direct technological applications. Discoveries are laws and principles that answer "What?" questions about natural phenomena underlying the production processes. Inventions, in contrast, are prescriptive in the sense that they provide answers to "How?" questions; inventions take the forms of blueprints and recipes. Table 2.1 presents this conceptual framework, and Table 2.2 remarks the two roles of useful knowledge studied in this dissertation. Other than Mokyr (2002), the role of the discovery-invention distinction and the usefulness of discoveries for inventive activity have been emphasized by Landes (1969), Rosenberg (1974), Nelson (1982) and Easterlin (1995) in explaining the history of technological progress. In one context, Weitzman (1998, p. 345) has suggested that knowledge accumulation has distinct *recombinant* and *productivity* aspects. The former corresponds, in a sense, to the role of discoveries for inventions. With reference to *knowledge capital*, Lucas (2002, p. 12) has asked "[w]hat can be gained by disaggregating into two or more knowledge-related variables." The model studied in this dissertation answers this question by showing that, when the productivity of inventive effort is endogenous to how large the stock of useful discoveries is, purposeful and costly invention may remain suboptimal. The distinction between discoveries and inventions is also emphasized by Haruyama (2009) in an endogenous growth model with perfectly competitive innovation. Howitt and Mayer-Foulkes (2005), O'Rourke et al. (2008) and Strulik (2009) are the

Table 2.2: The Usefulness of Knowledge: A Simple Framework

Discoveries	Inventions
useful in increasing the productivity of inventive effort	useful in increasing the productivity of worker hours

ones who have incorporated the distinction into the formal analysis of unified growth. However, the dual role of entrepreneurship for useful knowledge remains unexplored in a unified model with population growth and structural transformation. This is what this dissertation attempts to deliver.

At least since Schumpeter (1934), "entrepreneur-inventor"s are leading actors of the narratives of the Industrial Revolution. Solo (1951), Baumol (1990), Murphy et al. (1991) and Mokyr (2010), among others, have argued specifically that entrepreneurial invention was indeed the engine of technological progress during the first Industrial Revolution long before the rise of modern R & D lab. Peretto (1998b) has emphasized the distinction between "entrepreneurial invention" and corporate R & D in a second generation Schumpeterian model. Michelacci (2003) has previously studied the role of entrepreneurial skills in bringing inventions to markets. From another perspective, Doepke and Zilibotti (2008) and Galor and Michalopoulos (2009) have studied the role of entrepreneurial traits for long run economic development. This model, differently from all these, incorporates both occupational choice and entrepreneurial invention within a unified growth setting. Two-occupation framework of the model is similar to, and even simpler than, those of Lucas (1978), Murphy et al. (1991) and Michelacci (2003), and the formulation of entrepreneurial invention under perfect competition shares similarities with the treatments of Hellwig and Irmen (2001), Grossmann (2009) and Haruyama (2009).

The process of collective discovery by entrepreneurs has been described by Landes (2006, p. 6) as "the seventeenth-century European mania for tinkering and improving." Bekar and Lipsey (2004) go further to argue that the diffusion of Newtonian mechanics among British industrialists was the prime cause of the first Industrial Revolution. A similar argument on the diffusion of scientific culture, again with an emphasis on British success, is made by Jacob (1997). Kelly (2005) develops a network model that analyzes this type of collective learning for the industrial revolution. Lucas (2009) also emphasizes collective learning in a model that differentiates propositional knowledge from productivity. O'Rourke et al. (2008) and Milionis and Klasing (2009), with environments similar to that of Galor and Weil (2000), link the accumulation of propositional knowledge to human capital accumulation respectively through the number of high-skilled individuals and the individual-level stock of skills. Howitt and Mayer-Foulkes (2005) assume that the skill level of entrepreneurs is proportional to the average productivity associated with intermediate inputs of production. The last but not the least, Strulik (2009) suggests that propositional knowledge grows through learning-by-doing at the firm level. What differentiates this paper's formulation of propositional knowledge is the role of *the mass of entrepreneurs*. More entrepreneurs with longer lives create more useful discoveries given the quality of creating and diffusing these discoveries. This type of scale effect by which knowledge growth depends not on the mass of entire population but instead on a certain mass of urban population is emphasized more recently by Crafts and Mills (2009).

Chapter 3

The Model Economy

This chapter first introduces the model environment and then specifies market and ownership structures. Next follow the occupational structure and the decision problems solved by each occupational group. Market clearing conditions at the end close the model.

The model economy features some simplifications that are not uncommon in the unified growth literature: The economy is closed, there does not exist a political authority, and there is no physical capital. This is an economy of a single consumption good produced (i) with a traditional/rural technology to which land is an essential input and (ii) with a modern/urban technology that is utilized by entrepreneurs.

3.1 Environment

The calendar time of the model economy, denoted by t, is discrete with an infinite horizon: $t \in \mathbb{N}_+$.

3.1.1 Demographic Structure

There exist two overlapping generations: Individuals who are *adults* in period t give birth to *children* at the beginning of period t, i.e. at the beginning of their adulthood. Their surviving children become adults in period $t+1$.

Fertility

Reproduction is asexual, and $b_t \in \mathbb{R}_{++}$ denotes the total number of children a generic adult in period t optimally chooses to give birth to, i.e. the gross fertility per adult. Note that b_t, not being discrete in the model, represents the *average* gross fertility among period-t adults, and the total fertility rate of the economy is equal to $2b_t$ under the assumption that the sex ratio of population is 1/2.

Survival Probability

The survival of a child born at the beginning of period t is a Bernoulli event with the survival probability $s_t \in [0,1]$. For simplicity, (i) s_t is common across period-t children, (ii) s_t is known by period-t adults with certainty, and (iii) the sequence $\{s_t\}_{t\in\mathbb{N}_+}$ of survival probabilities is exogenous. Strongly motivated by actual data from developed economies, $\{s_t\}_{t\in\mathbb{N}_+}$ is assumed to be a non-decreasing sequence satisfying

$$s_0 < 1 \qquad \text{and} \qquad s^\star \equiv \lim_{t\to\infty} s_t = 1$$

At $t = 0$ that corresponds to some distant past, only some of the children born to an adult survive to adulthood. The distant future, on the other hand, represents an era at which the health/mortality revolution is complete so that all children survive.

Population Growth

Notice that, given b_t and s_t, the gross growth rate of adult population, i.e. the net fertility per adult, satisfies $n_t \equiv s_t b_t$. Denote by $N_t \in \mathbb{R}_{++}$ the adult population in period t and assume that $N_0 > 0$. The law of motion of N_t is

$$N_{t+1} = n_t N_t \tag{3.1}$$

Adult Longevity

Not less important than the number N_t of adult individuals is how long they live on average since labor is an input of the production and the invention technologies.[3] The simplest way of capturing the role of adult longevity within this two generations framework is to assume, as in Hazan and Zoabi (2006), that all period-t adults live a *fraction* $\ell_t \in [0,1]$ of period t. For simplicity, again, (i) ℓ_t is common across period-t adults, (ii) ℓ_t is known by period-t adults with certainty, and (iii) the sequence $\{\ell_t\}_{t\in\mathbb{N}_+}$ of longevity fractions is exogenous. Also motivated by actual data from developed economies, $\{\ell_t\}_{t\in\mathbb{N}_+}$ is assumed to be a non-decreasing sequence satisfying

$$\ell_0 < 1 \qquad \text{and} \qquad \ell^\star \equiv \lim_{t\to\infty} \ell_t = 1$$

The reasoning behind the survival probability thus applies to adult longevity: In some distant past, adult individuals do not live up to the maximum lifespan. When the health/mortality revolution is complete in distant future, however, they enjoy the longest life.

3.1.2 Endowments

Labor

Normalizing the length of a period to unity implies that period-t adults have a time endowment of ℓ_t *units* each. This time endowment is the only source of their homogeneous labor force. Children in contrast do not have a time endowment, and they remain idle until they become adults next period.

3. Adult longevity plays a key role for the timing of the industrial revolution in this model by affecting the optimality of inventive effort for any t.

Land

Land is a production factor of the traditional technology, and the total land endowment of the economy is normalized to unity. What fraction of individuals own land and at what proportions, on the other hand, are of minor importance from an analytical point of view since, as we shall see below, the model builds upon a very simplified view of what happens in the sector using the traditional technology as in Galor and Weil (2000) and Desmet and Parente (2009).

3.1.3 Preferences

A generic period-t adult derives lifetime utility from her consumption C_t and net fertility n_t.[4] The utility function representing these preferences is

$$U(C_t, n_t) \equiv C_t + \phi \ln(n_t) \qquad \phi > 0 \tag{3.2}$$

with boundary restrictions

$$C_t \geq \gamma > 0 \tag{3.3}$$

$$n_t \geq 1 \tag{3.4}$$

Strulik and Weisdorf (2008) and de la Croix and Licandro (2009) build on this type of non-homothetic preferences to understand the various aspects of the demographic transition. Here, a discussion of the notions embedded in (3.2)-(3.4) is necessary:

The preference parameter $\phi > 0$ determines the strength of parental desire towards reproduction. Such a parameter is a common element of economic models of demography.

The non-homotheticity of preferences is important for two reasons: First, the risk neutrality of preferences with respect to C_t eliminates the direct income effect on fertil-

4. Note that choosing n_t is identical to choosing b_t since s_t is known with certainty.

ity. This guarantees fertility to decrease with the cost of reproduction when the economy is sufficiently rich ($C_t > \gamma$). Fertility theories based on homothetic preferences face difficulties to generate a negative fertility-income relationship (Jones et al., 2011), and the model assumes away such difficulties, in part, with these non-homothetic preferences. The second reason of adapting risk neutral preferences with respect to C_t is to simplify the decision problem of entrepreneurs. As we shall see below, the decision toward entrepreneurial invention is in essence an expected utility problem with occupational choice.

The preference parameter $\gamma > 0$ in (3.3) denotes some baseline or "subsistence" level of adult consumption as in Galor and Weil (2000), Jones (2001) and others; γ is the level of consumption that partially determines the upper bound of fertility when the economy is sufficiently poor ($C_t = \gamma$). It shall be clear below that (3.3) simply reintroduces a direct income effect on fertility for a sufficiently poor economy: An adult's consumption is her first priority, and she has to adjust her fertility accordingly.[5]

The inequality in (3.4) represents the parental preference for *reproductive success* in transmitting genes to the next generations. What motivates this is the long run stability of population levels observed in nature. A successful parent is the one who leaves at least one surviving child.[6] (3.4) is a very simple way of introducing reproductive success in a model of fertility choice, but this suffices to produce the desired property:[7] The baseline level of net fertility is equal to unity as in Jones (2001), and this implies a stabilizing *level* of population when the economy is sufficiently rich.

5. In this model, γ represents subsistence not merely in *biological* terms; Landes (1969, pp. 13-14) and Voth (2003, p. 224) argue that Western European economies in general and England in particular were richer than many other economies around the world on the eve of the Industrial Revolution.

6. Recall that all surviving children become fecund at the beginning of their adulthood by construction.

7. de la Croix and Licandro (2009) and Strulik and Weisdorf (2011) incorporate reproductive success, respectively, into continuous and discrete time environments where parents choose not only the number of children they have but also the likelihood of these children's survival. This, in a sense, is a biological version of quantity-quality trade-off. With exogenous (s_t, ℓ_t), the model abstracts from this.

3.1.4 Technologies

Reproduction

There are two inputs of reproduction: Each child born, whether she survives or not, requires $\rho > 0$ units of time to be raised to adulthood and consumes $\psi > 0$ units of consumption good provided by her parent.[8]

Time cost of reproduction is a common theme of a vast majority of fertility theories, and it plays an essential role in this model of demographic transition. Simply, the cost of reproduction increases in an economy that records sustained increases in labor productivity since a unit of time becomes more expensive.

Goods cost of reproduction, albeit not necessary for fertility to decline in advanced stages of economic development, adds to the consistency of the model in which adults must consume a minimum amount of $\gamma > 0$.

The Modern Technology

The following mention of firms and entrepreneurs is necessary to introduce the remaining technologies of the model with maximum clarity: In any competitive equilibrium of this model economy, (i) there exists a set $[0, E_t]$ of firms that operate with the modern technology of production, and (ii) each firm using the modern technology is owned by an entrepreneur. Hence, the set of entrepreneurs is also $[0, E_t]$ where $E_t < N_t$ is the endogenously determined mass of entrepreneurs.[9]

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8. In contrast with existing models featuring a time cost of reproduction, this model does not restrict the time spent on reproduction to be the parent's time exclusively. We shall see below that this interpretation is necessary for a particular simplifying assumption on the cost of reproduction for entrepreneurs.

9. Also note that, since there exists a unique consumption good in this economy, the traditional technology might be sufficiently productive, compared to the modern technology, to imply $E_t = 0$ at least initially. Since the purpose here is to study the role of entrepreneurship, the rest of the analysis implicitly restricts (i) the model's fixed parameters, (ii) the exogenous (state) variables, and (iii) the initial values of endogenous state variables such that $E_t > 0$ for all t.

Production: Let $i \in [0, E_t]$ be a representative firm. The Cobb-Douglas technology of production is

$$Y_{it} = \left(X_{it} h_{wit}\right)^{\lambda} h_{mit}^{1-\lambda} \qquad \lambda \in (0,1) \tag{3.5}$$

where Y_{it} denotes output, $X_{it} > 0$ denotes labor productivity associated with worker hours h_{wit}, and h_{mit} denotes management hours allocated to production.

Recall that the labor endowment of (adult) individuals is homogeneous. Accordingly, what differentiates worker and management hours in (3.5) is only the nature of the tasks in the question: Two distinct tasks are required to produce the good. Workers are the ones who actually produce the good in its finalized form with their eye-hand coordination, and managers are the ones who tell workers *what* to do and *how* to do it.

The knowledge content of management is represented by X_{it} which is a measure of prescriptive knowledge; X_{it} identifies the "quality" of the production process.

Productivity Change: What cause the labor productivity of the modern technology to grow in time are new inventions created through costly inventive projects. The invention projects, requiring research hours directed to invention, generate a stochastic number of inventions, and each new invention increases a baseline level of productivity by some fixed factor.

Suppose, as in Desmet and Parente (2009) and many others, that the modern sector firms in period t have access to the average productivity $\overline{X}_t \in \mathbb{R}_{++}$ attained by the firms of the previous generation. The term *access* here refers to the intergenerational diffusion of prescriptive knowledge embedded in $\overline{X}_t$. Simply define this average as

$$\overline{X}_t \equiv E_{t-1}^{-1} \int_0^{E_{t-1}} X_{jt-1} \mathrm{d}j \tag{3.6}$$

where j indexes period-$(t-1)$ firms.

Define now firm i's operating productivity as

$$X_{it} \equiv \sigma^{z_{it}} \overline{X}_t \tag{3.7}$$

where $\sigma > 1$ is the stepsize of inventions and $z_{it} \in \mathbb{N}_+$ is the stochastic number of inventions satisfying

$$z_{it} \sim \text{Pois}\left(a_{it}\right) \tag{3.8}$$

In (3.8), $a_{it} \in \mathbb{R}_+$ denotes the arrival rate of inventions and satisfies

$$a_{it} = \theta\xi\left(K_t\right) h_{rit} \qquad \theta > 0 \tag{3.9}$$

where h_{rit} denotes the amount of hours allocated to research by firm i. This invention technology features constant returns to scale with respect to rival labor input h_{rit}.

The novelty here is the term $\theta\xi\left(K_t\right)$, i.e. the level of *research productivity* per unit of inventive effort, where $K_t \in \mathbb{R}_{++}$ denotes the number of costlessly accessible useful discoveries to be utilized in the process of inventing. $\xi\left(K_t\right)$, assumed to be continuously differentiable, is the intrinsic component of research productivity explained by *epistemic foundations*; $\theta > 0$ is the extrinsic productivity of invention.

Three restrictions on $\xi\left(K_t\right)$ are central to the main results of this paper, and it is more convenient for the sake of the discussion to formally state them first:

$$\xi'(K) > 0 \text{ for all } K < \infty \qquad \xi(0) = 0 \qquad \lim_{K\to\infty} \xi(K) = 1 \tag{3.10}$$

The first restriction here represents the conjecture that it is more likely to be successful in invention if more useful discoveries are available. In light of the earlier discussion about the role of propositional knowledge, this is a plausible way to restrict the epistemic function $\xi(K_t)$.[10]

The second restriction imposes that discoveries are essential inputs of invention projects. In Mokyr's (2002, pp. 13-14) words, "[t]he likelihood that a laptop computer would be developed in a society with no knowledge of computer science, advanced electronics, materials science, and whatever else is involved is nil."

The third restriction originates from the conjecture that the productivity of research per unit of inventive effort should have an upper limit when $K_t \to \infty$ as in, e.g., Weitzman (1998): The arrival rate is a *growth* factor of productivity, and this has to have an upper limit naturally imposed by (neuro)physiological capabilities of humans; the arrival rate should be bounded above for any level of inventive effort that is less than infinity. That K_t goes to infinity means, on the other hand, that every single *knowable* thing about the natural phenomena underlying the production processes is known. This is the ultimate enlightenment that eliminates what causes a unit of inventive effort to be less productive than its full potential of θ, i.e. the narrowness of the epistemic bases for invention. An inventor, knowing practically everything about the natural phenomena, does not have to spend her time with trying to realize which certain discoveries are useful. She simply generates an expected number of inventions with constant (maximum) productivity θ as in many other endogenous technology models.

The Traditional Technology

Production: The Cobb-Douglas production function that determines the volume Y_{Tt} of output produced with the traditional technology is

$$Y_{Tt} = \left(\tilde{X}_{Tt} H_{Tt}\right)^{\eta} L_{Tt}^{1-\eta} \qquad \eta \in (0,1) \tag{3.11}$$

10. Clearly, the invention technology (3.9) does not incorporate the fishing out effect: Research productivity per unit of inventive effort does not decrease with $\overline{X}_t$. Chapter 7 discusses an alternative postulation where research productivity increases with K_t and decreases with $\overline{X}_t$.

where $\tilde{X}_{Tt}$ is labor productivity, H_{Tt} is the total amount of hours allocated to production, and L_{Tt} is land input.

Productivity Change: Since the main purpose is not to develop a theory of traditional technology's labor productivity growth, a very simple law of motion for $\tilde{X}_{Tt}$ is adapted following Desmet and Parente (2009):

$$\frac{\tilde{X}_{Tt+1}}{\tilde{X}_{Tt}} = \max\left\{\zeta_0, \left(\frac{\overline{X}_{t+1}}{\overline{X}_t}\right)^{\zeta_1}\right\} \qquad \zeta_0 > 1, \zeta_1 > 0 \tag{3.12}$$

In (3.12), $\zeta_0 > 0$ denotes an exogenous (gross) growth rate of $\tilde{X}_{Tt}$. The second argument of the maximum function imposes that $\tilde{X}_{Tt}$ grows faster than ζ_0 if the spillover effect from the modern technology is high enough. (3.12) thus captures initially slow and stable and then accelerating rates of productivity growth of the land-based technologies.

Collective Discovery

The last technology to be specified is of the stock K_t of useful discoveries. In its generic form, what creates new discoveries is simply time. The conjecture, as introduced earlier, is that entrepreneurs, owning the firms using the modern technology, collectively discover new pieces of propositional knowledge during their lifetime. They not only create new knowledge in this serendipitous way individually but also share what they create with each other in their common environment, e.g. in coffeehouses. This is a network effect that is consistent with the common knowledge characterization of useful discoveries.

The simplest way to formalize this is a linear knowledge production function of the form

$$K_{t+1} - K_t = \omega \ell_t E_t \qquad \omega > 0 \tag{3.13}$$

where $\ell_t E_t$ denotes the total lifetime of all entrepreneurs that, interpreted as a single variable, is the input of the collective discovery process and $\omega > 0$ represents the quality of environment for creating and sharing useful discoveries. This constant thus represents geographical, cultural and social determinants of collective discovery process.[11]

3.2 Occupations

Recall that the set of entrepreneurs is $[0, E_t]$ in equilibrium, and i indexes entrepreneurs. The remaining mass $N_t - E_t$ of adult individuals forms the set of workers.

Equilibria of this model build upon two decision problems solved respectively by entrepreneurs and workers. The purpose of this and the following sections is to derive these decision problems.

The problems involve fertility choice for both occupations and an additional decision regarding the inventive activity for entrepreneurs. These decisions are finalized in the sense that they embed other decisions these individuals have to take optimally. That is, if the occupation is chosen optimally and the decision problems are solved, both at the beginning of adulthood, other actions follow contingently.

To proceed with maximum clarity, it is necessary at this stage to briefly discuss what entrepreneurs and workers do.

3.2.1 Entrepreneurs

An entrepreneur, as noted earlier, establishes a firm that uses *the modern technology*. There are no establishment costs involved. Once an entrepreneur establishes the firm, she becomes the business owner until the end of her life. The firm dies with its entrepreneur, and each generation raises new entrepreneurs who establish and own new firms using the modern technology.

As the business owner, an entrepreneur purchases an optimal level of worker hours. The other necessary input, i.e. management, is provided by the entrepreneur herself.

11. Note that the linearity with respect to $\ell_t E_t$ is not a crucial assumption of the model. The qualitative nature of results does not change as long as $K_{t+1} - K_t$ is an increasing function of $\ell_t E_t$.

This creates an incentive for the entrepreneur to spend resources for inventive activities.

An entrepreneur, as any adult individual, also chooses her fertility. The key simplifying assumption here is that entrepreneurs hire workers to take care of their children.[12] From a technical point of view, this simplifying assumption is necessary to separate the choice of fertility from the decision regarding the inventive activity for an entrepreneur.

3.2.2 Workers

Those who choose to become workers supply hours for three activities:

First, some workers are employed by entrepreneurs to work in their firms using the modern technology.

Second, some workers, again being employed by entrepreneurs, take care of entrepreneurs' children at home.

Finally, some workers choose to work in the firms that use the traditional technology.

3.3 Markets

All sellers and all buyers in any market of this economy are price-takers. This is a model of *perfect competition*.

Two things are traded in markets. First, the consumption good is demanded by all adult individuals in the economy and produced/sold by firms that use either the traditional or the modern technology. The consumption good is the numéraire of the economy. Second, a worker hour is traded at wage $W_t > 0$.

12. One justification for this assumption, despite its sexist tone, is this: In the British inventors data set of Meisenzahl and Mokyr (forthcoming), we observe only one woman out of 759 individuals. If it is not entirely misleading to imagine that mothers take care of their children and a business is run by one person only, then the cost of fertility for a family where the husband chooses to be an entrepreneur must be foregone worker earnings.

3.4 Production Sectors

3.4.1 Modern Sector

Entrepreneurial invention occurs, if it is optimal, in the modern sector and generates positive economic growth through productivity increases. Then, the question is what creates an incentive to invent for each entrepreneur under perfect competition. That each entrepreneur manages her firm by providing the manager hours herself answers this question: Management is basically the quasi-fixed factor of production that implies positive profit for the entrepreneur. Assuming that the invention technology is costlessly accessible, it shall be the case that the entrepreneur may find it optimal to allocate some of her scarce time endowment to inventive activity.

Optimal Ex Post Profit

The first task in characterizing what happens in the modern sector is to derive the optimal *ex post* profit maximized through the optimal choice of the demand for worker hours.

Ignoring the complications regarding the timing of events *within* a period and following, e.g., Aghion and Howitt (2009), assume that an entrepreneur can choose the level of optimal demand for worker hours h_{wit} for given (X_{it}, h_{mit}, W_t) to maximize profit. The profit function is defined as

$$\Pi\left(h_{wit}, X_{it}, h_{mit}, W_t\right) \equiv \left(X_{it} h_{wit}\right)^{\lambda} h_{mit}^{1-\lambda} - W_t h_{wit} \tag{3.14}$$

The optimal h_{wit} and $\Pi\left(h_{wit}, X_{it}, h_{mit}, W_t\right)$ thus satisfy

$$h_{wit} \equiv \arg\max_{h} \Pi\left(h, X_{it}, h_{mit}, W_t\right) = \frac{\lambda^{\frac{1}{1-\lambda}} X_{it}^{\frac{\lambda}{1-\lambda}} h_{mit}}{W_t^{\frac{1}{1-\lambda}}} \tag{3.15}$$

$$\Pi_{it} = (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} \left(\frac{X_{it}}{W_t}\right)^{\frac{\lambda}{1-\lambda}} h_{mit} \tag{3.16}$$

Entrepreneurial Invention

Replace X_{it} with $\sigma^{z_{it}}\overline{X}_t$. The invention technology, via (3.8) and (3.9), then implies the expected profit as in

$$\mathbf{E}\Pi_{it} \equiv \sum_{z=0}^{\infty}\left[\frac{a_{it}^{z}\exp\left(-a_{it}\right)}{z!}\right]\left[(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{\sigma^{z}\overline{X}_t}{W_t}\right)^{\frac{\lambda}{1-\lambda}}h_{mit}\right] \tag{3.17}$$

The term in the first brackets denotes the Poisson probability of generating z inventions given a_{it}. The term in the second brackets is the level of optimal *ex post* profit when entrepreneur generates z inventions given a_{it}.

What does not allow entrepreneur i to be able to spend an infinite amount of resources to inventive activity is her time constraint: $h_{mit}+h_{rit}\leq\ell_t$. Since $\mathbf{E}\Pi_{it}$ is strictly increasing in h_{mit}, this constraint holds with strict equality in equilibrium:

$$h_{mit}+h_{rit}=\ell_t \tag{3.18}$$

Together with (3.9), (3.18) allows us to rewrite the expected profit as

$$\mathbf{E}\Pi_{it} = \sum_{z=0}^{\infty}\left[\frac{a_{it}^{z}\exp\left(-a_{it}\right)}{z!}\right]\left[(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{\sigma^{z}\overline{X}_t}{W_t}\right)^{\frac{\lambda}{1-\lambda}}\left(\ell_t-\frac{a_{it}}{\theta\xi\left(K_t\right)}\right)\right] \tag{3.17$'$}$$

Lemma 1: The expected profit in (3.17′) can be rewritten as

$$\mathbf{E}\Pi_{it} = \exp\left(\Sigma a_{it}\right)\Lambda\left(\frac{\overline{X}_t}{W_t}\right)^{\Gamma}\left(\ell_t-\frac{a_{it}}{\theta\xi\left(K_t\right)}\right) \tag{3.17$''$}$$

where

$$\Gamma\equiv\frac{\lambda}{1-\lambda} \qquad \Lambda\equiv(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} \qquad \Sigma\equiv\sigma^{\frac{\lambda}{1-\lambda}}-1$$

are defined for notational ease.

Proof— See Appendix A. ■

(3.17″) now identifies the cost of and the return to inventive activity. The first term, $\exp\left(\Sigma a_{it}\right)$, increases the expected profit by some factor greater than one depending on the size σ of invention and the productivity elasticity λ of output. The last term in the last parentheses, $a_{it}/\theta\xi\left(K_t\right)$, is the time cost of inventing with an expected number a_{it} of inventions. Clearly, (3.17″) specifies the *deterministic* level of profit when no inventive activity is undertaken ($a_{it}=0$). X_{it} is equal to $\overline{X}_t$ in this case.

Production

The production at the firm level is implied by (3.5) given $(h_{wit}, X_{it}, h_{mit})$ where h_{wit} satisfies (3.15). The final task to complete the discussion of the modern sector is thus to define the total *ex post* production:

$$Y_t = \int_0^{E_t} Y_{it} \mathrm{d}i \tag{3.19}$$

3.4.2 Traditional Sector

There exists a single firm in this sector, and there do not exist property rights over the land.[13] With total land endowment of the economy normalized to unity, these assumptions allow us to work with the following restricted form of the traditional technology

$$Y_{Tt} = X_{Tt} H_{Tt}^{\eta} \tag{3.12′}$$

where $X_{Tt} \equiv \tilde{X}_{Tt}^{\eta}$.

Those who supply worker hours to this sector's firm are assumed to be the owners

13. The alternative assumption of a unit mass continuum of identical firms does not alter the results under perfect competition given the constant-returns-to-scale technology (3.12).

of the firm with *equal* ownership shares. Hence, these workers equally appropriate a positive profit in addition to their income originating from labor supply. It turns out that workers in the traditional sector earn, *per unit hour*, the average product $X_{Tt}H_{Tt}^{\eta-1}$. In any equilibrium, then, we must have

$$X_{Tt}H_{Tt}^{\eta-1} = W_t \tag{3.20}$$

due to the perfect mobility of labor across sectors.

From (3.20) follows the mass N_{Tt} of workers employed in the traditional sector:

$$N_{Tt} \equiv \frac{H_{Tt}}{\ell_t} \tag{3.21}$$

Note that a richer treatment with land rents would pose no serious difficulties given the constant-returns-to-scale technology (3.12) and under, e.g., *the law of primogeniture* that simplifies the intergenerational transmission of land ownership. In that case, those who own, e.g., equal shares of land would appropriate the land rents in every generation, leaving the land ownership to one of their surviving children.

3.5 Decision Problems

This subsection finally derives the decision problems solved by entrepreneurs and workers. Three remarks follow before proceeding to the derivations:

First, notice that workers are indifferent between taking care of their children on their own and hiring workers as entrepreneurs do so. In what follows, it is assumed that workers take care of children with their own time.

Second, given preferences and the entire discussion above, the new elements needed are the budget constraints individuals face.

Finally, note that only two problems are of interest, one for workers and one for entrepreneurs. For workers, there is a unique problem because all earn W_t per unit hour.

For entrepreneurs, the uniqueness of the problem follows from the fact that they all face the same baseline productivity $\overline{X}_t$. That there exists only one problem solved by all entrepreneurs in turn implies that entrepreneurs of generation t act symmetrically with respect to certain choice variables. Then, once the uncertainty regarding the inventive activity is resolved, a cross-section probability distribution characterizes the remaining entrepreneur-level variables.

3.5.1 Workers' Problem

Rewrite the utility function (3.2) for our representative worker who gives birth to b_{wt} children and consumes C_{wt} units:

$$U_{wt} \equiv C_{wt} + \phi \ln \left(s_t b_{wt} \right)$$

Since this worker earns the unit wage of W_t and spends ρb_{wt} amount of worker hours and ψb_{wt} amount of goods for child care, her budget constraint reads

$$C_{wt} + \psi b_{wt} \leq (\ell_t - \rho b_{wt}) W_t$$

That U_{wt} is strictly increasing in C_{wt} leads this budget constraint to hold with strict equality. Eliminating b_{wt} via $n_{wt} = s_t b_{wt}$ then implies

$$C_{wt} = \left[\ell_t - \rho \left(\frac{n_{wt}}{s_t} \right) \right] W_t - \psi \left(\frac{n_{wt}}{s_t} \right) \tag{3.22}$$

Note that $C_{wt} \geq \gamma$, following from (3.3), and (3.22) now determine the upper level of n_{wt} as in

$$n_{wt} \in \left[1, \frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi} \right] \tag{3.23}$$

and workers, choosing n_{wt}, seek to maximize

$$U_{wt} = \left[\ell_t - \rho\left(\frac{n_{wt}}{s_t}\right)\right] W_t - \psi\left(\frac{n_{wt}}{s_t}\right) + \phi \ln\left(n_{wt}\right) \tag{3.24}$$

subject to (3.23).

3.5.2 Entrepreneurs' Problem

The utility function (3.2) can be rewritten for entrepreneur i as

$$U_{it} \equiv C_{it} + \phi \ln\left(s_t b_{it}\right)$$

where C_{it} and b_{it} respectively denote her consumption level and the number of children she gives birth to. The budget constraint she faces reads

$$C_{it} + \psi b_{it} + \rho b_{it} W_t = \Pi_{it} \tag{3.25}$$

where $\psi b_{it} + \rho b_{it} W_t$ indicates the total cost of having b_{it} children.

As in the derivation of the workers' problem, U_{it} is strictly increasing in C_{it}, implying that the budget constraint above holds with strict equality. This allows us to rewrite U_{it} as

$$U_{it} = \Pi_{it} - \psi b_{it} - \rho b_{it} W_t + \phi \ln\left(s_t b_{it}\right)$$

The only worth-to-be-emphasized difference with the workers' problem here is that entrepreneur i's decision towards inventive activity is taken under uncertainty, i.e. to maximize $\mathbf{E}U_{it} = \mathbf{E}\Pi_{it} - \psi b_{it} - \rho b_{it} W_t + \phi \ln\left(s_t b_{it}\right)$. Thus, the appropriate form of the minimum consumption constraint for entrepreneurs' problem is

$$\mathbf{E}C_{it} \geq \gamma$$

This inequality, the budget constraint above and $\mathbf{E}\Pi_{it}$ as in (3.17″) imply that net fertility $n_{it} = s_t b_{it}$ is bounded above as in

$$n_{it} \in \left[1, \frac{\left[\exp\left(\Sigma a_{it}\right) \Lambda \left(\frac{\overline{X}_t}{W_t}\right)^{\Gamma} \left(\ell_t - \frac{a_{it}}{\theta \xi (K_t)}\right) - \gamma\right] s_t}{\rho W_t + \psi}\right] \tag{3.26}$$

Therefore, entrepreneurs, choosing n_{it} and a_{it}, seek to maximize

$$\begin{aligned} \mathbf{E} U_{it} = \exp\left(\Sigma a_{it}\right) \Lambda \left(\frac{\overline{X}_t}{W_t}\right)^{\Gamma} \left(\ell_t - \frac{a_{it}}{\theta \xi \left(K_t\right)}\right) \\ - \psi \left(\frac{n_{it}}{s_t}\right) - \rho \left(\frac{n_{it}}{s_t}\right) W_t + \phi \ln\left(n_{it}\right) \end{aligned} \tag{3.27}$$

subject to (3.26) and

$$a_{it} \in \left[0, a_t^{\max}\right] \tag{3.28}$$

where $a_t^{\max} \equiv \theta \xi \left(K_t\right) \ell_t > 0$ is the maximum possible arrival rate when entrepreneur spends her entire labor endowment to inventive activity.

3.5.3 Occupational Choice

By solving the decision problems derived above, workers and entrepreneurs act optimally. What completes the occupational choice with individual rationality is therefore the following *equal utilities* restriction:

$$\mathbf{E} U_{it} = U_{wt} \tag{3.29}$$

An adult individual, taking all actions at the beginning of her adulthood, must be indifferent between becoming an entrepreneur and becoming a worker.[14]

14. Note that, within this simple two-occupation framework with ex ante identical adult individuals, the equilibrium mass of entrepreneurs is determined in the labor market residually; see below.

3.6 Market Clearing Conditions

The final task is to close the model through the market clearing conditions:

The market for worker hours clears via

$$(N_t - E_t - N_{Tt})\,\ell_t - (N_t - E_t)\,\rho b_{wt} = \int_0^{E_t} h_{wit}\mathrm{d}i + E_t \rho b_{it} \tag{3.30}$$

where the R.H.S. and the L.H.S. of (3.30) respectively denote the total demand for and the total supply of worker hours. The first term in the R.H.S. is the total amount of hours employed in the modern sector firms, and the second term in the R.H.S. denotes the amount of hours allocated to entrepreneurs' child care. The supply of worker hours is determined by the total available hours $(N_t - E_t - N_{Tt})\,\ell_t$ minus hours not supplied to the market by workers due to child care.

The market for the consumption good clears via

$$Y_{Tt} + Y_t = \int_0^{E_t} (C_{it} + \psi b_{it})\,\mathrm{d}i + (N_t - E_t)\,(C_{wt} + \psi b_{wt})$$

where the R.H.S. denotes the total supply from traditional and modern sectors and the L.H.S. denotes the total demand by entrepreneurs and workers.

Chapter 4

Static, Dynamic, and Asymptotic Equilibria

This chapter defines and analyzes the equilibria of the model economy. The main purpose in this chapter is to establish the analytical foundations of the model economy's equilibrium path from some initial period to the infinite future. Naturally, for things to be interesting from a unified growth perspective, this equilibrium path should be long enough to cover the transition from stagnation to growth in its entirety.

The static general equilibrium of the model is defined first. The important properties of this unique static general equilibrium are studied next. The dynamic general equilibrium of the model is then defined as a sequence of static general equilibria. The unique dynamic general equilibrium is further restricted to characterize the asymptotic equilibrium of the model for $t \to \infty$. A discussion on the asymptotic equilibrium and its stability properties leads the chapter to a concluding section on the model economy's equilibrium path.

4.1 Static General Equilibrium

Definition 1: A *static general equilibrium* of this model, for any $t \in \mathbb{N}_+$, is a collection $\{n_{wt}, C_{wt}, H_{Tt}, N_{Tt}, Y_{Tt}, E_t, \{n_{it}, a_{it}, h_{rit}, h_{mit}, z_{it}, X_{it}, h_{wit}, Y_{it}, \Pi_{it}, C_{it}\}_{i \in [0,E_t]}, Y_t\}$ of quantities and the wage W_t such that, given the state vector $(s_t, \ell_t, N_t, K_t, \overline{X}_t, X_{Tt})$,

- n_{wt} solves the workers' problem characterized by (3.23) and (3.24),
- (n_{it}, a_{it}) solves the entrepreneurs' problem characterized by (3.26)-(3.28),

- all adult individuals are indifferent between becoming an entrepreneur and becoming a worker through (3.29),

- the market for worker hours clears through (3.30),[15] and

- (3.5), (3.7), (3.8), (3.9), (3.15), (3.16), (3.18), (3.19), (3.12′), (3.20), (3.21), (3.22) and (3.25) are satisfied. ■

Proposition 1: There exists a unique static general equilibrium (SGE).

Proof — See Appendix A. ■

4.1.1 Ex Ante Symmetry and Ex Post Heterogeneity

Since all entrepreneurs face the unique vector $(W_t, \overline{X}_t, K_t, s_t, \ell_t)$ of given variables in utility maximization, they act symmetrically regarding the inventive activity:[16]

$$a_{it} = a_t \quad \text{and} \quad h_{rit} = h_{rt} \quad \text{for all } i \in [0, E_t]$$

Notice that ex ante symmetry across entrepreneurs translates into ex post heterogeneity because of the stochastic nature of invention. Despite spending an equal amount of research hours to inventive activity, entrepreneurs do not generate an equal number of inventions. A fraction of them record no inventions, a fraction only one, and another fraction two, and so on. Clearly, these fractions in cross-section establish a Poisson distribution under the assumption that E_t is a large number. Specifically, for any arrival rate $a > 0$, the ex ante probability of generating z inventions is equal to the ex post fraction of entrepreneurs with z inventions.

15. Note that the market for the consumption good clears via Walras' Law. Hence, the market clearing condition for this market is not included in equilibrium defining equations.

16. That inventive effort is common across entrepreneurs in turn implies that they all spend an equal amount h_{mt} on management.

The Poisson fractions naturally determine the variation of the operating productivity X_{it}, of the demand for worker hours h_{wit}, of the volume of production Y_{it}, of the level of profit Π_{it} and of the level of consumption C_{it} across entrepreneurs. All these distributions are identical to the unique (stationary) Poisson distribution Pois(a_t).

4.1.2 The Invention Threshold and the Industrial Revolution

The unique SGE of the model is characterized either by $a_t = 0$ or by $a_t \in (0, a_t^{\max})$, and *the industrial revolution in the model* is defined as the endogenously occurring switch from the equilibrium regime of $a_t = 0$ to that of $a_t > 0$.

The threshold property of inventive activity directly follows from the solution to the entrepreneurs' problem through a rather straightforward algebra involving an application of Kuhn-Tucker Theorem. Intuitively, the marginal cost of increasing the expected number of inventions from zero to an infinitesimally small amount is a strictly positive number that may well exceed its marginal return. This is due to the fact that entrepreneur has to decrease her management input to increase her inventive effort. That is, the return to and the cost of invention is not additively separable; see (3.17″). Formally, we have the following:

Proposition 2: The unique SGE of the model is characterized by an invention threshold such that

$$a_t = \begin{cases} 0 & \text{if } \xi\left(K_t\right)\ell_t < \left[\theta\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)\right]^{-1} \\ \theta\xi\left(K_t\right)\ell_t - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1} & \text{otherwise} \end{cases} \tag{4.1}$$

Proof— See Appendix A. ■

Notice that ℓ_t, σ and λ increase the return to inventive activity and that $\theta\xi\left(K_t\right)$ decreases the cost of it. Thus, these four determinants of inventive activity create threshold effects accordingly.

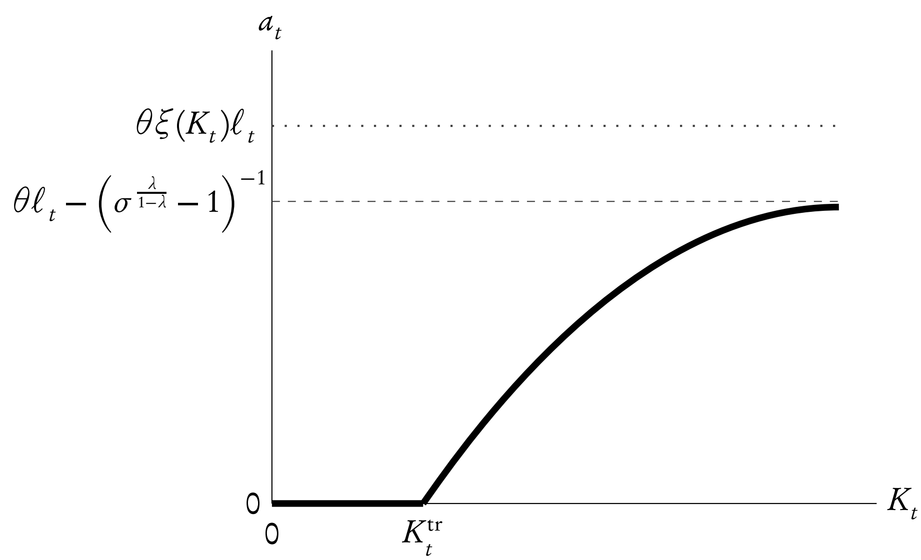

Figure 4.1: The Arrival Rate as a Function of the Stock of Discoveries

For the rest of the analysis, it is useful to approach this threshold property by taking K_t as the endogenous state variable that determines whether $a_t = 0$ or $a_t > 0$. Suppose that the inverse function $\xi^{-1}(\bullet)$ exists, and define the time-variant threshold originating from Proposition 2 as

$$K_t^{\text{tr}} \equiv \xi^{-1}\left[\theta^{-1}\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1}\ell_t^{-1}\right] \tag{4.2}$$

Since $\xi^{-1}(\bullet)$ is strictly increasing, as implied by strictly increasing $\xi(K)$ for all $K < \infty$, K_t^{tr} is strictly decreasing in θ, σ, λ and ℓ_t as expected. Figure 4.1 draws a_t as a function of K_t where K_t^{tr} is as defined above.

From Proposition 2 follows the optimal allocation of entrepreneurs' time to management. When invention is not optimal, entrepreneurs clearly spend their entire labor endowment ℓ_t to routine management to fully benefit from the profit opportunities under constant productivity. Formally, we have

$$h_{mt} = \begin{cases} \ell_t & \text{if } K_t < K_t^{\text{tr}} \\ \left[\theta\xi\left(K_t\right)\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)\right]^{-1} & \text{otherwise} \end{cases} \tag{4.3}$$

4.1.3 Productivity, Wage, and Output

The level of the real wage W_t at the unique SGE, along with other things, determines the level of economic development in this economy. It specifically determines the size of the traditional sector and the optimal level of fertility. Thus, it is necessary to understand how the real wage itself is tied to productivity.

The mapping from productivity to wage in the modern sector of the economy is

$$W_t = (1-\lambda)^{1-\lambda}\lambda^{\lambda}\delta\left(a_t, K_t, \ell_t\right)\overline{X}_t^{\lambda} \tag{4.4}$$

where $\delta\left(a_t, K_t, \ell_t\right)$ is an auxiliary function defined as in

$$\delta\left(a_t, K_t, \ell_t\right) \equiv \exp\left[(1-\lambda)\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)a_t\right]\left(1 - \frac{a_t}{\theta\xi\left(K_t\right)\ell_t}\right)^{1-\lambda}$$

When inventive activity is not optimal ($a_t = 0$), we have $\delta(0, \bullet, \bullet) = 1$ implying that $W_t = (1-\lambda)^{1-\lambda}\lambda^{\lambda}\overline{X}_t^{\lambda}$. This simply corresponds to the unit price of a worker hour that would prevail in a competitive model of occupational choice with Cobb-Douglas technology *and without entrepreneurial invention*.

When inventive activity is optimal ($a_t > 0$), management input is tied to invention technology through the optimal use of entrepreneurs' time. W_t in competitive equilibrium thus embeds this effect via $\delta(\bullet, \bullet, \bullet)$ function. Notice, however, that the only source of sustained growth in W_t is still the sustained growth of $\overline{X}_t$; a_t and $\delta\left(a_t, K_t, \ell_t\right)$ are both bounded above for any t.

Returning to the volume of output produced in the modern sector, the unique SGE is characterized by

$$Y_t \equiv \int_0^{E_t} Y_{it}\mathrm{d}i = E_t\left(\frac{\lambda}{1-\lambda}\right)^{\lambda}\overline{X}_t^{\lambda}\ell_t\delta\left(a_t, K_t, \ell_t\right) \tag{4.5}$$

The total volume of output is proportional to the number of firms/entrepreneurs simply because ex post heterogeneity obeys the well-behaved Poisson distribution.

4.1.4 The Structural Transformation

To see how the structural transformation is determined in this model, consider the labor and the output shares of the modern sector defined as in

$$f_t^{NM} \equiv 1 - \frac{N_{Tt}}{N_t} \qquad \text{and} \qquad f_t^{YM} \equiv \frac{Y_t}{Y_{Tt} + Y_t}$$

In the unique SGE, the labor share N_{Tt}/N_t of the traditional sector, measured in individuals (not in worker hours), satisfies

$$\frac{N_{Tt}}{N_t} = \left(\frac{1}{\ell_t N_t}\right)\left(\frac{X_{Tt}}{W_t}\right)^{\frac{1}{1-\eta}} \tag{4.6}$$

and the total volume of output produced in the traditional sector reads

$$Y_{Tt} = \frac{X_{Tt}^{\frac{1}{1-\eta}}}{\left(W_t\right)^{\frac{\eta}{1-\eta}}} \tag{4.7}$$

Recalling that the traditional and the modern sectors produce the same good, a higher level of traditional sector productivity X_{Tt} implies a higher labor share N_{Tt}/N_t of this sector in contrast to the dual economy models in which (i) the land-based technology (or agriculture) is used to produce *food* and (ii) there exists a minimum food consumption restriction.

Population growth and longevity gains, on the other hand, decrease N_{Tt}/N_t via the *push effect* of the larger supply of hours $\ell_t N_t$ which is due to the dependency on fixed land input.

The *pull effect* of the modern sector's productivity is represented by W_t. Figure 4.2 pictures the labor share of the traditional sector as a function of the real wage.

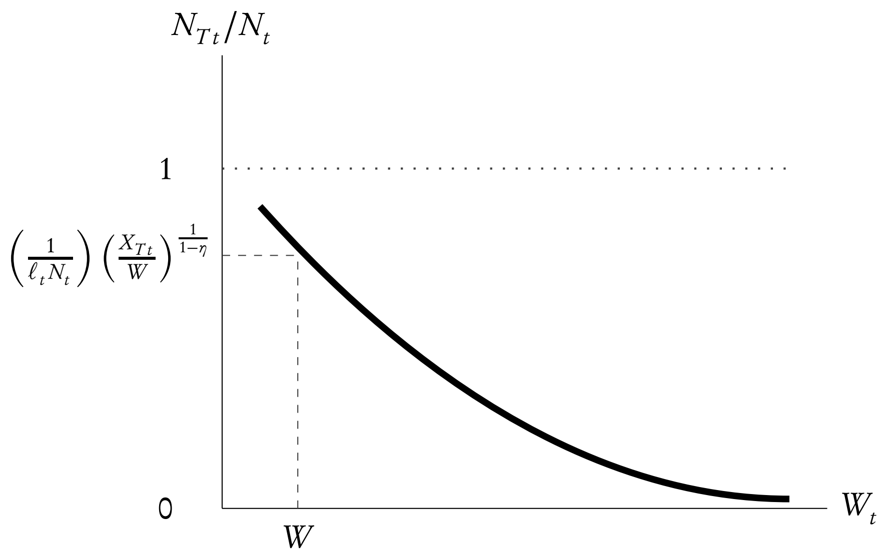

Figure 4.2: The Labor Share of the Traditional Sector as a Function of Real Wage

4.1.5 Fertility and the Demographic Transition

The unique SGE of the model captures the phases of a demographic transition mainly depending on the evolution of real wage W_t.

Proposition 3: The unique SGE is characterized by identical fertility choice among all adults, satisfying $n_{wt} = n_{it} = n_t$, such that

$$n_t = \begin{cases} \frac{(W_t \ell_t - \gamma)s_t}{\rho W_t + \psi} & \text{if } W_t < \frac{\phi+\gamma}{\ell_t} \\ \frac{\phi s_t}{\rho W_t + \psi} & \text{if } W_t \in \left[\frac{\phi+\gamma}{\ell_t}, \frac{\phi s_t - \psi}{\rho}\right] \\ 1 & \text{if } W_t > \frac{\phi s_t - \psi}{\rho} \end{cases} \tag{4.8}$$

where the total births per adult, denoted by b_t, is given by n_t/s_t.

Proof— See Appendix A. ■

(4.8) identifies the thresholds for fertility determined simply by the cost of and the return to reproduction. For sufficiently low levels of potential lifetime income $W_t \ell_t$, adult individuals' priority remains as their own consumption. When this consumption constraint binds, increasing W_t and ℓ_t allow them to increase their optimal net fertility basically because they have more resources to spend on reproduction. Not surprisingly,

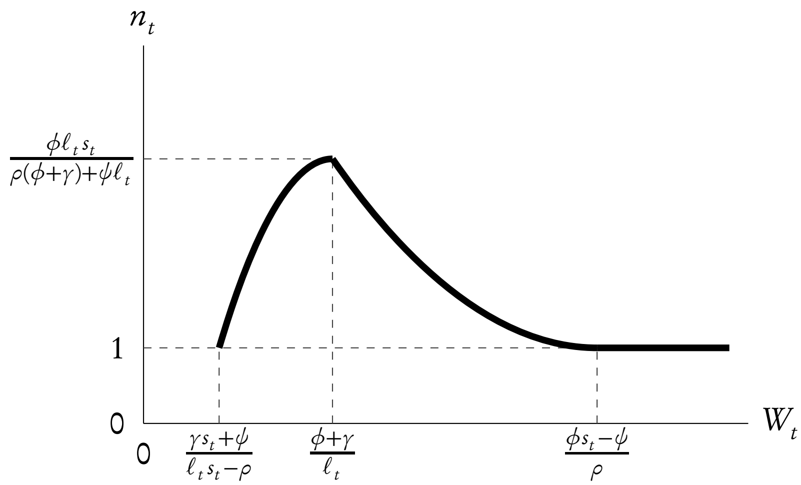

Figure 4.3: Net Fertility as a Function of Real Wage

the parental desire for reproduction does not affect fertility during this first phase of the demographic transition.

If the potential lifetime income continues to increase, it eventually becomes optimal to decrease fertility since adults are now free to choose an optimal consumption higher than the "subsistence" level of γ. The income effect vanishes completely at this second stage of the demographic transition, and both net and gross fertility decline with increasing W_t. This phase of the demographic transition is thus characterized by the vanishing role of adult longevity. The substitution effect, on the other hand, still implies a negative effect of W_t on fertility. Maintaining a level of net fertility that is greater than unity becomes sufficiently costly for adults at the advanced stages of economic development, i.e. $W_t > (\phi s_t - \psi)\rho$. Thus, the third stage of the demographic transition records net fertility equal to unity so that the level of adult population stabilizes. Net fertility as a function of the real wage is shown in Figure 4.3.

4.1.6 The Supply of Entrepreneurship

The supply of entrepreneurship is central to the equilibrium path of the model economy since the growth rate of K_t is a function of the mass E_t of entrepreneurs. This mass satisfies

$$E_t = (1-\lambda)\left(1 - \frac{N_{Tt}}{N_t} - \frac{\rho b_t}{\ell_t}\right) N_t \tag{4.9}$$

Note that the mass of entrepreneurs satisfies (4.9) regardless of the invention threshold and the thresholds of net fertility. There exist simply three determinants of the supply of entrepreneurship: First, out of N_t adult individuals, N_{Tt} individuals work in the primitive sector. Second, a fraction of the total labor endowment is allocated to fertility per adult. Finally, $(1-\lambda)$ is the ratio of entrepreneurs among the mass of individuals who choose to work in the modern sector.

4.1.7 Output per worker and Output per capita

Two measures of the standard of living are output per worker and output per capita, and it is useful to define them before proceeding to the analysis of the dynamic general equilibrium.

Since every adult has b_t children, the total population is equal to $P_t \equiv (1+b_t)N_t$. Thus, output per worker y_t^{pw} and output per capita y_t^{pc} read

$$y_t^{pw} = \frac{Y_{Tt} + Y_t}{N_t} \qquad\qquad y_t^{pc} = \frac{Y_{Tt} + Y_t}{(1+b_t)N_t} \tag{4.10}$$

where Y_{Tt} and Y_t respectively satisfy (4.7) and (4.5).

4.2 Dynamic General Equilibrium

To define the dynamic general equilibrium, it is necessary to specify how the vector $(s_t, \ell_t, N_t, K_t, \overline{X}_t, X_{Tt})$ of state variables evolves from t to $t+1$. For the exogenous state variables, recall that the sequences $\{s_t, \ell_t\}_{t\in\mathbb{N}_+}$ are exogenously given. Next, adult population N_t and the stock of discoveries K_t evolve respectively with (3.1) and (3.13). Thus, the laws of motion to recover are those of $\overline{X}_t$ and X_{Tt}.

To derive the former, iterate (3.6) to obtain

$$\overline{X}_{t+1} = E_t^{-1} \int_0^{E_t} X_{it} \mathrm{d}i$$

Substituting X_{it} with $\sigma^{z_{it}} \overline{X}_t$ and noting once again that the *ex post* fraction of entrepreneurs with z inventions is equal to the *ex ante* probability of generating z inventions result in

$$\overline{X}_{t+1} = \overline{X}_t \sum_{z=0}^{\infty} \left(\frac{a_t^z \exp(-a_t)}{z!} \right) \sigma^z$$

This law of motion in turn reduces into the following after some arrangements as in the proof of Lemma 1:

$$\overline{X}_{t+1} = \overline{X}_t \exp\left[(\sigma - 1) a_t\right] \tag{4.11}$$

Thus, as in Aghion and Howitt (1992) and others, the growth rate of (average) productivity is explained by the size σ and the intensity a_t of inventions.

Using (3.12) and the law of motion derived above, the growth rate of $X_{Tt} = \tilde{X}_{Tt}^{\eta}$ can be written as

$$\frac{X_{Tt+1}}{X_{Tt}} = \max\left\{\zeta_0^{\eta}, \exp\left[\eta \zeta_1 (\sigma - 1) a_t\right]\right\} \tag{4.12}$$

Definition 2: A *dynamic general equilibrium* of this model economy is a sequence of static general equilibria, existing for all $t \in \mathbb{N}_+$, together with the sequences $\{N_t, K_t, \overline{X}_t, X_{Tt}\}_{t \in \mathbb{N}_{++}}$, that satisfies the laws of motion (3.1), (3.13), (4.11) and (4.12) given the sequences $\{s_t, \ell_t\}_{t \in \mathbb{N}_+}$ and the initial values $(N_0, K_0, \overline{X}_0, X_{T0})$. ■

Proposition 4: There exists a unique dynamic general equilibrium (DGE).

Proof— See Appendix A. ■

4.3 The Asymptotic Equilibrium

The asymptotic equilibrium of the model economy is the (unique) limiting SGE for $t \to \infty$. This limit, as argued earlier, corresponds to some distant future in historical time. Motivated by growth empirics and by the "spirit" of the model, the asymptotic equilibrium to be constructed and analyzed is the one with

- the positive growth of modern sector productivity and output,
- the constant level of population, and
- the declining traditional sector

Notice that, in this asymptotic equilibrium, output per worker and output per capita exhibits exponential growth. One should however be careful about the nature of this result: By definition, the model economy does not reach its asymptotic equilibrium in finite time. The asymptotic equilibrium is conceptually and technically different from a steady-state equilibrium if the latter is defined as a static equilibrium that is reached in finite time and in which the bounded variables of the model remain constant from some finite t to the infinite future. This sort of steadiness is satisfied by some variables of the model in the unique DGE. Net fertility n_t, for example, reaches its baseline level of unity in finite time if the real wage grows to hit an endogenous threshold. Some other key variables, however, do not reach their asymptotic values. The arrival rate a_t, for example, converges to

$$a^{\star} \equiv \lim_{t\to\infty} a_t = \theta \lim_{t\to\infty} \xi(K_t) \lim_{t\to\infty} \ell_t - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1}$$

which reduces into $a^{\star} = \theta - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)$ with $\xi(K_t) \to 1$ and $\ell_t \to 1$. The former limit condition, as specified earlier, requires $K_t \to \infty$.

4.3.1 Two Preliminary Tasks

To construct the asymptotic equilibrium, two preliminary tasks have to be completed: First, we need to preclude the uninteresting DGE where an industrial revolution is not even *possible*. Next, we need to specify the condition(s) under which the traditional sector does not operate when $t \to \infty$.

The Possibility of an Industrial Revolution

The earlier restrictions that we put on $\xi(K)$ function in (3.10) do not ensure $K_t^{tr} < \infty$; see (4.2). In what follows, it is assumed that the inverse function $\xi^{-1}(\bullet)$ and its argument $\theta^{-1}\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1}\ell_t^{-1}$, for all t, are such that $K_t^{tr} < \infty$.

This assumption is less restrictive than it may sound because the simplest forms of $\xi(K)$ function that satisfy (3.10) also imply that $K_t^{tr} < \infty$ for sufficiently high θ, σ or λ given any ℓ_t.

The Decline of the Traditional Sector

For the DGE with an industrial revolution, what would ensure the decline of the traditional sector is the slower growth of X_{Tt} than that of W_t for $t \to \infty$:

$$f^{NM\star} \equiv \lim_{t\to\infty} f_t^{NM} = 1 - \left(\frac{1}{\ell^\star N^\star}\right) \lim_{t\to\infty} \left[\left(\frac{X_{Tt}}{W_t}\right)^{\frac{1}{1-\eta}}\right]$$

If the spillover from the modern to the traditional sector, i.e. from $\overline{X}_t$ to X_{Tt}, never becomes active in the unique DGE, then $\zeta_0^\eta < \exp\left[\lambda(\sigma-1)a^\star\right]$ would be necessary as dictated by $W_t \propto \overline{X}_t^\lambda$, (4.11) and (4.12). On the other hand, if the spillovers become active at some $t < \infty$, what ensures the slower growth of X_{Tt} compared to that of W_t for $t \to \infty$ is

$$\zeta_1 < \frac{\lambda}{\eta} \tag{4.13}$$

In what follows, it is assumed that (4.13) is satisfied. The former condition of $\zeta_0^\eta < \exp\left[\lambda(\sigma-1)a^\star\right]$, on the other hand, shall be of no importance since the value of ζ_0 that explains the long epochs of stagnation before the industrial revolution is always sufficiently low to activate the sectoral spillover from $\overline{X}_t$ to X_{Tt}; see below.

Table 4.1: Equilibrium Regimes

Regime	Invention	Fertility	Spillover from $\overline{X}_t$ to X_{Tt}
I	$a_t = 0$	n_t incr. w/ W_t	inactive
II	$a_t = 0$	n_t decr. w/ W_t	inactive
III	$a_t = 0$	$n_t = 1$	inactive
IV	$a_t > 0$	n_t incr. w/ W_t	inactive
V	$a_t > 0$	n_t decr. w/ W_t	inactive
VI	$a_t > 0$	$n_t = 1$	inactive
VII	$a_t = 0$	n_t incr. w/ W_t	active
VIII	$a_t = 0$	n_t decr. w/ W_t	active
IX	$a_t = 0$	$n_t = 1$	active
X	$a_t > 0$	n_t incr. w/ W_t	active
XI	$a_t > 0$	n_t decr. w/ W_t	active
XII	$a_t > 0$	$n_t = 1$	active

4.3.2 The Global Stability

Given the exogenous sequences $\{s_t, \ell_t\}_{t\in\mathbb{N}_+}$, the model economy's unique asymptotic equilibrium characterized above is *globally stable* for the set of initial values and for the set of model parameters that satisfy $E_t > 0$ for all $t \in \mathbb{N}_+$.

The difficulty here is that the dynamical system of the model is not simple enough to allow us to rewrite it as an autonomous system of normalized variables. Instead, the analysis can only be carried out through a conditional dynamical system and with the help of a phase diagram. What follows is a discussion of why the model's asymptotic equilibrium is globally stable, and the formal analysis of the conditional dynamical system is presented in Appendix B due to its tediousness.

Recall that there exist two regimes of invention, three regimes of fertility, and two regimes of the growth of the traditional sector productivity X_{Tt}. Hence there exist twelve possible equilibrium regimes from which the model economy can start its evolution at $t = 0$; see Table 4.1. The task is to understand why the economy eventually enters Regime XII and stays in this regime for $t \to \infty$.

The first key to understand why the economy remains in a regime of $a_t > 0$ for large enough t is the *inevitability* of an industrial revolution:

Proposition 5: Let the unique DGE feature $a_0 = 0$. Then, there exists a period $0 < t^{tr} < \infty$ such that $a_{t^{tr}-1} = 0$ and $a_{t^{tr}} > 0$. That is, if the economy starts its evolution in a period at which invention is not optimal, an industrial revolution inevitably starts at some future period.

Proof— See Appendix A. ■

Notice that the inevitability result of Proposition 5 implies that the arrival rate a_t of invention remains positive for $t > t^{tr}$. That ℓ_t is a non-decreasing sequence by assumption and that K_t^{tr} is decreasing in ℓ_t imply K_t remains greater than K_t^{tr} for $t > t^{tr}$. This in turn implies that the arrival rate a_t of inventions increases to $a^\star$ after the industrial revolution. Directly following is thus the positive asymptotic growth of $\overline{X}_t$ and, thus, of W_t. Clearly, then, the asymptotic equilibrium does not exist in Regimes I, II, III, VII, VIII, and IX.

The perpetual growth of W_t is the main driver of the demographic transition. Regardless of the stage of the demographic transition at $t = 0$, net fertility n_t eventually becomes equal to its baseline level of unity at some $t < \infty$, i.e. $n^\star = 1$. This in turn implies that the level of adult population stabilizes at $N^\star > 0$. Hence, the asymptotic equilibrium does not exist in Regimes IV, V, X, and XI as well.

Finally, the asymptotic growth of $\overline{X}_t$ and W_t eliminates Regime VI for $t \to \infty$. As stated earlier and explained below, ζ_0 is sufficiently low such that, at some $t < \infty$, the growth rate $\overline{X}_{t+1}/\overline{X}_t$ becomes sufficiently high to activate the sectoral spillovers; the growth rate X_{Tt+1}/X_{Tt} exceeds ζ_0^η.

4.3.3 The Asymptotic Rates of (Economic) Growth

Define the gross growth rate of the stock K_t of useful discoveries using (3.13) and (4.9) as

$$g_{Kt} \equiv \frac{K_{t+1}}{K_t} = 1 + \frac{\omega \ell_t \left[(1-\lambda) \left(1 - \frac{N_{Tt}}{N_t} - \frac{\rho b_t}{\ell_t} \right) N_t \right]}{K_t} \tag{4.14}$$

With $n^\star = s^\star = 1$, gross fertility per adult attains the limit value $b^\star = 1$ in finite time.

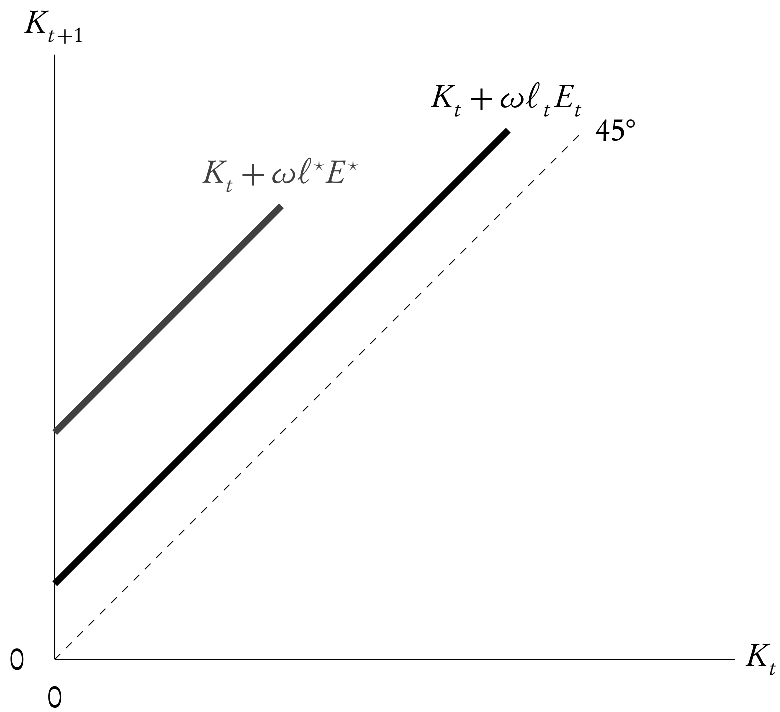

Figure 4.4: The Process of Collective Discovery for $t < \infty$ and for $t \to \infty$

Since N_{Tt} converges to zero, g_{Kt} satisfies

$$g_K^\star \equiv \lim_{t\to\infty} g_{Kt} = 1 + \frac{\omega(1-\lambda)(1-\rho)N^\star}{\lim_{t\to\infty} K_t}$$

where $(E^\star \equiv)(1-\lambda)(1-\rho)N^\star$ is the asymptotic value of the mass E_t of entrepreneurs. Clearly, $E^\star > 0$ implies both $g_{Kt} > 1$ for $t < \infty$ and $g_K^\star = 1$ for $t \to \infty$. These dynamics are pictured in Figure 4.4.

Let $g_{\overline{X}t} \equiv \overline{X}_{t+1}/\overline{X}_t$ denote the gross growth rate of the modern sector productivity. Using the results derived earlier, we can write this growth rate as

$$g_{Xt} = \exp\left[(\sigma-1)\left(\theta\xi(K_t)\ell_t - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1}\right)\right]$$

In the limit, then, the asymptotic growth rate $g_X^\star$ is a function of technological parameters θ, σ, and λ, all having definite interpretations.

Returning to output per worker (y_t^{pw}) and output per capita (y_t^{pc}), first note that the asymptotic growth of these variables is explained entirely by the growth of $\overline{X}_t$: For $t \to \infty$, the traditional sector declines, and adult population and fertility are fixed. Recalling that the modern sector output is proportional to $\overline{X}_t^{\lambda}$, we can conclude that the asymptotic growth rate $g_y^\star$ of living standards is equal to

$$g_y^\star = \exp\left[\lambda(\sigma-1)\left(\theta - \left(\sigma^{\frac{\lambda}{1-\lambda}}-1\right)^{-1}\right)\right] = \left(g_X^\star\right)^{\lambda}$$

In the asymptotic equilibrium, then, output per capita and output per worker grow at a rate *proportional to* but *lower than* that of labor productivity where the stock of useful discoveries and the level of population do not exhibit growth. Formally, we have

$$g_X^\star > g_y^\star > 1 \qquad g_K^\star = g_N^\star = 1$$

4.4 The Absence of the Weak and the Strong Scale Effects

An important feature of the model economy is to be noted before proceeding to the construction of the equilibrium path: Not surprisingly, the weak and the strong scale effects of population size are absent in this economy. In fact, the absence of scale effects on output per worker and output per capita holds not only at the asymptotic equilibrium but also along the transition.

To see why there does not exist a weak scale effect, recall that (i) the volume Y_{Tt} of the traditional sector output is not related with adult population N_t, and (ii) the volume Y_t of the modern sector output is proportional to E_t (which itself is proportional to N_t). That is, if the level of population is higher, there exists a larger mass of entrepreneurs, *ceteris paribus*, and, *in the absence of any direct positive externality of E_t or N_t on total output*, the SGE levels of y_t^{pc} and y_t^{pw} are not affected positively by N_t.

The strong scale effect, on the other hand, is absent because of the following: Even though the model does not explicitly incorporate the horizontal dimension of technological progress (or product innovation) as in the second-generation Schumpeterian models of Young (1998), Peretto (1998a), Aghion and Howitt (1998, Ch. 12) and Dinopoulos and Thompson (1998), that the innovative sector of the economy is inhabited by a mass E_t of independently innovating firms still implies that the total research effort of the economy is spread thinly across modern sector firms. Given that there does not exist a direct link from population level to productivity growth, the transition and the asymptotic growth rates of y_t^{pc} and y_t^{pw} do not change with the level of population.[17]

4.5 The Equilibrium Path: From Stagnation to Growth

We can now construct an equilibrium path of the model economy from some distant past $t = 0$ to some distant future $t \to \infty$. Specifically, the interest here is on an equilibrium path with an initial SGE that exists in Regime I. This is clearly the most interesting equilibrium path from a unified growth perspective. The stock of useful discoveries is small enough to make invention not optimal, productivity increases only in the traditional sector and at a constant minuscule rate, and the level of the modern sector productivity is low enough to make fertility an increasing function of the real wage.

There surely exists a set of initial values of state variables that further characterize the initial SGE in Regime I as follows:

- net fertility and the rate of population growth are low,
- gross fertility is high,
- the traditional sector's labor and output shares are large, and
- the levels of the standards of living are low.

The model inputs, i.e. structural parameters, initial values, and exogenous variables, must be such that this initial equilibrium is sustained from $t = 0$ to some future period under quasi-statis.

17. Strictly speaking, there exists a type of strong scale effect on the growth of K_t. This is discussed below in Section 7.5.

First, realistically suppose that s_t and ℓ_t are constant respectively at their historically lowest levels s_0 and ℓ_0 for a long episode of history until the industrial revolution is near. This returns a level of net fertility which is equal to

$$n_0 = \frac{(W_0\ell_0 - \gamma)s_0}{\rho W_0 + \psi}$$

Notice that, for sufficiently low W_0, net fertility and the rate of population growth are sufficiently low. Next, recalling that the labor share of the traditional sector is equal to

$$\frac{N_{Tt}}{N_t} = \left(\frac{1}{\ell_0 N_t}\right)\left(\frac{X_{Tt}}{W_0}\right)^{\frac{1}{1-\eta}}$$

a restriction needs to be put on the growth of X_{Tt} to imply a quasi-static level of N_{Tt}/N_t. It turns out that if ζ_0 satisfies

$$\zeta_0^{\frac{\eta}{1-\eta}} = n_0$$

the labor share N_{Tt}/N_t remains stable at its historically high level for long periods; slowly increasing population requires slowly increasing traditional sector productivity at this equilibrium.

The model thus explains the stagnation of output per worker y_t^{pw} and output per capita y_t^{pc} at their historically lowest (constant) levels y_0^{pw} and output per capita y_0^{pc} if ζ_0 satisfies the above condition, but the critical level of ζ_0 is not arbitrary: If there is a stable quasi-static equilibrium of sectoral shares of labor and output, this equilibrium *has to* be characterized by stagnation of living standards.[18]

An important remark is in order: The initial quasi-static stagnation equilibrium just characterized is not *strongly Malthusian* since the real wage that determines the level of optimal fertility changes with the modern sector's productivity but not with the level of

18. Regarding the knife-edge type of restriction that is put on ζ_0, note that endogenizing the productivity growth of the traditional technology via learning-by-doing as in Strulik and Weisdorf (2008) would imply an automatically stabilizing dynamical system at the stagnation equilibrium just characterized.

population. Increasing population in this model does not decrease the real wage. This strong Malthusian link would be constructed by allowing the modern and the traditional sector wages to differ in a richer framework of rural-urban migration.

Returning now to the equilibrium path, stagnation in strict sense ends when the economy eventually hits its invention threshold. Collective discovery, operating all along during the stagnation, eventually closes the knowledge gap. The real wage starts increasing, leading to increasing gross and net fertility. Increasing inventive effort initiates the sectoral spillover from the modern to the traditional sector at some period, but faster population growth and technological progress in the modern sector still imply declining output and labor shares of the traditional sector. Once the modern sector productivity is sufficiently large, gross fertility starts declining because adults' time is now sufficiently expensive. Decreasing population growth and still increasing inventive effort leads even faster growth of output per capita/worker. On the other hand, urbanization and industrialization continue. At some advanced stage of economic development with a sufficiently high modern sector productivity, net fertility becomes equal to its baseline level of unity because the adult individual now finds it optimal to sustain reproductive success at its limit. The stabilizing population in turn implies that the growth rate of the stock of useful discoveries slows down. Even though the existing entrepreneurs keep discovering new knowledge, each generation's contribution gets marginally smaller compared to the existing stock of discoveries. Once this slowing down starts, the *increase* of the inventive effort starts decelerating. With constant population and constant fertility, the growth of modern sector productivity converges to its maximum. In the future, according to this model without material resource constraints other than that of labor, humanity faces no limits in increasing prosperity.

Chapter 5

The Timing of the Industrial Revolution: Some Tentative Results

The consensus view of world economic history emphasizes the lateness of modern economic growth: The Industrial Revolution in England started around 50000 years later than the rise of modern human populations, i.e. populations that share cultural universals such as language, art, religion, and toolmaking. If one rescales the history of modern human populations to 365 days, the Industrial Revolution would have had occurred around 44 hours ago.

The model constructed and analyzed above suggests that living standards may stagnate for several millennia if purposeful invention in the modern sector is too costly, i.e. if lives remain sufficiently short and the stock of useful discoveries remains sufficiently small.

This chapter presents some tentative analytical results on the timing of the industrial revolution in the model. The timing question basically asks which factors are most essential in explaining the length of time passed with stagnating standards of living in a given economy for some fixed initial period.

From a methodological point of view, some may find the timing question very ambitious. This is due, in part, to the simplicity of the models that assume away several *a priori* important aspects of the transition from stagnation to growth as the present model does and, in part, to the complexity of the phenomenon being dealt with. Economic historian Jones (2010, p. 245), e.g., claims that

> There is no determinate solution to the puzzle of why the industrial revolution took place, and where and when it did so. All that can be achieved is a narrowing of the range of possible mixes.

The model, by providing answers to why the industrial revolution took place and why that late, narrows the range of possible mixes in Jones' (2010) terms. Yet, answering questions such as "Why England, but not France or China?" and "Why 18th century, but not the 14th?" is at best harder.

First of all, the timing question may not be simply separable from the location question because a mechanism that explains the timing of the industrial revolution for a given economy may provide insights to understand the absence of an industrial revolution in another economy. As in Mokyr (2002), indeed, the collective discovery by entrepreneurs not only explains the lateness of the industrial revolution in England but also implies that China, with its infamously large bureaucracy and very low urbanization, did not benefit from collective learning because of a lack of entrepreneurship.

Second, as suggested long ago by Crafts (1977), *luck* may have played a key role to put England in front of France and other Western European nations for a few generations in industrialization. The model-based cross-sectional calibration results of Voigtländer and Voth (2006) indeed imply that France had a much higher likelihood of industrialization than China on the eve of Industrial Revolution. A valid argument thus can be made to reformulate the location question as "Why Western Europe, but not China?"

Yet, there exists another difficulty about how pre-industrial economies should be compared with regard to the timing of the industrial revolution. Pomeranz (2000), e.g., argues that, due to the regional diversity of big geographical areas such as Western Europe and China, the comparisons should be made between economies of appropriate size. England, e.g., should be compared not with China as a whole but its most economically advanced Lower Yangzi region. In light of these difficulties, the tentative analytical results on the timing of the industrial revolution should better be read as suggestive, not conclusive.

5.1 Preliminaries

Since we direct our attention to the DGE with $a_0 = 0$, it is useful for future reference to note that the modern sector productivity $\overline{X}_t$ and the real wage W_t stagnate respectively at

$$\overline{X}_0 > 0 \qquad W_0 = (1-\lambda)^{1-\lambda}\lambda^{\lambda}\overline{X}_0^{\lambda} > 0$$

Clearly, W_0 is increasing in λ, for any $\overline{X}_0 > 0$, if $\lambda \in (0.5, 1)$.

5.1.1 First-Order and Second-Order Effects

One difficulty associated with the timing question is that the period t^{tr} at which the industrial revolution starts does not have a closed-form solution in terms of the model's parameters and state variables. Thus, the feasible analysis here is to question whether a higher value of a relevant model parameter or of an exogenous state variable, *ceteris paribus*, delays or hastens the start of the industrial revolution *from the perspective of some given period*. That is, we fix a period $t < t^{tr}$ and ask, for each element of the vector

$$(s_t, \ell_t, \gamma, \psi, \phi, \rho, \eta, \theta, \sigma, \lambda, \omega),$$

whether the industrial revolution would be delayed or hastened if the chosen element of the vector is higher than some benchmark value *given* the endogenous state variables. These are labeled as *first-order* effects.

The *second-order* effects, remaining analytically implicit, are the ones that run through the endogenous state variables. That is, when a model parameter is higher, it affects not only the SGE of any fixed period $t < t^{tr}$ but also the SGE of $t+1$, $t+2$, and so on. As we shall see in the next chapter, these second-order effects are indeed substantial especially through the population growth.

5.1.2 Threshold vs. Growth Effects

There are two types of first-order effects to be distinguished regarding the timing question: First, there exists an endogenous and time-varying threshold K_t^{tr} that defines the knowledge gap

$$K_t - K_t^{tr} = K_t - \xi^{-1}\left[\theta^{-1}\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1} \ell_t^{-1}\right]$$

for any given K_t. Clearly, for the inevitable industrial revolution to start, this gap should be greater than or equal to zero at t^{tr}. Hence, the threshold effect is related with the question of *how far away the industrial revolution is* for some $t < t^{tr}$.

The growth effect, on the other hand, is related with the question of *how fast the economy moves towards its invention threshold* to decrease its knowledge gap, and the (gross) growth rate g_{Kt} of K_t determines this speed for any given K_t^{tr}. Rewrite this growth rate by substituting N_{Tt}/N_t and then W_0 as

$$g_{Kt} = 1 + \omega \ell_t (1-\lambda)\left[1 - \left(\frac{1}{\ell_t N_t}\right)\left(\frac{X_{Tt}}{(1-\lambda)^{1-\lambda}\lambda^{\lambda}\overline{X}_0^{\lambda}}\right)^{\frac{1}{1-\eta}} - \frac{\rho b_t}{\ell_t}\right]\left(\frac{N_t}{K_t}\right) \tag{5.1}$$

Notice that g_{Kt} changes with the level of optimal gross fertility b_t. This separates the analysis further into two cases, i.e. the start of the industrial revolution (i) in the regime where fertility is increasing with the real wage and (ii) in the regime where fertility is decreasing with the real wage.[19] Therefore, we should rewrite g_{Kt} explicitly for each case. For the regime where fertility is increasing with the real wage, (5.1) becomes

$$g_{Kt} = 1 + \omega(1-\lambda)\left[\frac{\gamma\rho + \psi\ell_t}{\rho W_0 + \psi} - \left(\frac{1}{N_t}\right)\left(\frac{X_{Tt}}{W_0}\right)^{\frac{1}{1-\eta}}\right]\left(\frac{N_t}{K_t}\right) \tag{5.1'}$$

19. The regimes in which $n_t = 1$ are of minor importance for the present analysis since, for the economy to be in such a regime, the real wage must already be sufficiently high.

Table 5.1: The Timing of the Industrial Revolution: Case 1 (n_t incr. w/ W_t)

Experiment	Threshold Effect	Growth Effect
Higher s_t	No Effect	No Effect
Higher ℓ_t	Hastening	Hastening
Higher γ	No Effect	Hastening
Higher ψ	No Effect	Hastening
Higher ϕ	No Effect	No Effect
Higher ρ	No Effect	Delaying
Higher η	No Effect	Hastening
Higher θ	Hastening	No Effect
Higher σ	Hastening	No Effect
Higher λ	Hastening	Ambiguous
Higher ω	No Effect	Hastening

and, for the regime where fertility is decreasing with the real wage, we have

$$g_{Kt} = 1 + \omega(1-\lambda)\left[\frac{\ell_t(\rho W_0 + \psi) - \rho\phi}{\rho W_0 + \psi} - \left(\frac{1}{N_t}\right)\left(\frac{X_{Tt}}{W_0}\right)^{\frac{1}{1-\eta}}\right]\left(\frac{N_t}{K_t}\right) \qquad (5.1'')$$

5.2 Results

The partial derivatives of K_t^{tr} and g_{Kt} with respect to model parameters and exogenous variables identify the threshold and the growth effects given the set $(N_t, K_t, \overline{X}_t, X_{Tt})$ of endogenous state variables.

Tables 5.1 and 5.2 summarize the results of the analysis. "No Effect" entries indicate a partial derivative of zero. An "Ambiguous" entry on the other hand indicates a partial derivative with an ambiguous sign.

In Tables 5.1 and 5.2, the threshold effects are identical and simply follow from strictly increasing $\xi^{-1}(\bullet)$. As discussed earlier, ℓ_t, σ and λ increase the return to inventive activity, and θ decreases the cost of it. The remaining model parameters and s_t do not generate threshold effects.

Returning to the growth effects, first note that s_t does not generate growth effects, neither in Case 1 nor in Case 2. This follows from the fact that gross fertility does not

Table 5.2: The Timing of the Industrial Revolution: Case 2 (n_t decr. w/ W_t)

Experiment	Threshold Effect	Growth Effect
Higher s_t	No Effect	No Effect
Higher ℓ_t	Hastening	Hastening
Higher γ	No Effect	No Effect
Higher ψ	No Effect	Hastening
Higher ϕ	No Effect	Delaying
Higher ρ	No Effect	Delaying
Higher η	No Effect	Hastening
Higher θ	Hastening	No Effect
Higher σ	Hastening	No Effect
Higher λ	Hastening	Ambiguous
Higher ω	No Effect	Hastening

change with the survival probability in this model.[20]

A higher value of adult longevity ℓ_t has the hastening growth effect in both cases. Entrepreneurs generate a larger number of useful discoveries if they live longer.

A higher value of ω, implying a higher "quality" of the process of collective discovery, also hastens the industrial revolution in both Case 1 and Case 2.

A larger labor exponent η of the traditional technology implies a lower share of the traditional sector and has the hastening growth effect in both cases.

More interesting are the effects of the parameters that determine gross fertility. The unit good cost of reproduction ψ decreases gross fertility in both cases. Accordingly, a higher value of ψ has hastening effects in both cases because a smaller volume of labor endowment is allocated to child rearing due to the decreasing level of fertility. The unit time cost parameter ρ, however, have the reverse effect. Even if a higher value of ρ decreases gross fertility by increasing the cost of reproduction, it also *directly* increases the total amount of hours allocated to child rearing. ρ generates a delaying growth effect in both cases since the direct effect dominates.

A higher value of γ implies a lower level of gross fertility by tightening the budget constraint of adults in the regime where fertility is increasing with the real wage. This

20. Notice that s_t does have a second-order (hastening) growth effect through N_t.

is a hastening growth effect, but γ does not generate a growth effect in Case 2. Instead, a growth effect in Case 2 is generated by ϕ. If adults prefer to have more children, this translates into a delaying effect, again, by increasing the total amount of hours allocated to child rearing.

The set of parameters that do not generate growth effects include θ and σ. Both are technological parameters governing the return to the inventive effort, but they do not affect the growth rate of K_t in any regime before the industrial revolution.

The labor exponent λ of the modern sector technology, in addition to its hastening threshold effect, creates a growth effect. The direction of this growth effect however remains ambiguous. In both Case 1 and Case 2, a higher value of λ may imply either a lower or a higher mass of entrepreneurs because, technically, both fertility and the labor share of the traditional sector change with W_0 and hence with λ. For any given level of W_0, a higher value of λ would imply a lower mass of entrepreneurs since the level of optimal demand for worker hours by entrepreneurs is larger on average. Yet, since W_0 changes with λ, we have

$$\frac{\partial g_{Kt}}{\partial \lambda} \lesseqgtr 0$$

in both Case 1 and Case 2.

Chapter 6

The Quantitative Analysis

This chapter presents a quantitative analysis of the model economy. The purpose of this analysis is threefold: First of all, a quantitative analysis of the model, disciplined empirically via formal calibration, is essential to evaluate the model's success in explaining the phases of economic development from pre-industrial times to the contemporary era. Second, since the very long-run evolution of the model economy is of interest, this chapter presents the simulations of the calibrated model economy and studies the transition to its previously described asymptotic equilibrium. Finally, the timing question is revisited quantitatively, and some counter-factual experiments, again based on the calibrated model, are implemented to conclude which factors are most important in determining the timing of the industrial revolution. These experiments are necessary since the tentative results presented in the last chapter do not reveal the overall timing effects of model inputs.

6.1 Preliminaries

The empirical counterpart of the model economy is that of England since it is the first industrialized economy in the world. Fortunately, England is also the economy of which the largest data set on certain variables of interest is available compared to the other Western European economies.

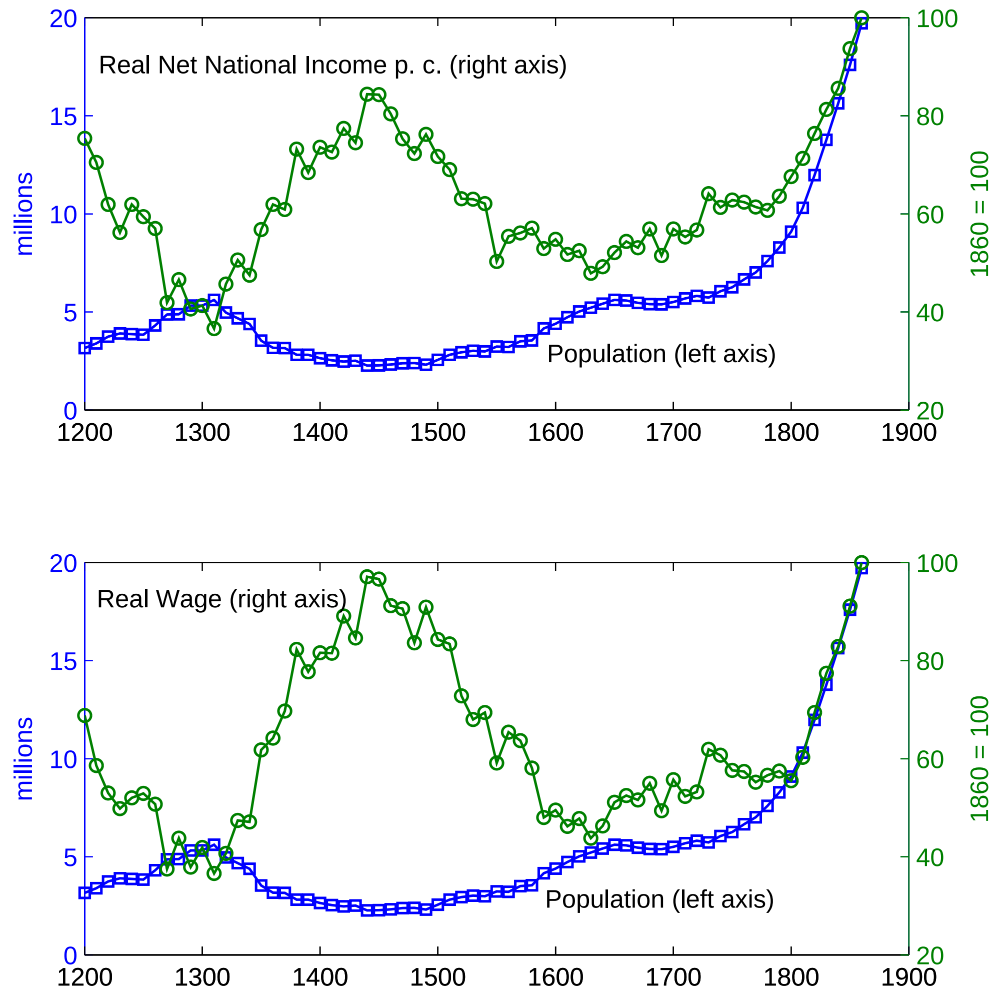

Figure 6.1: Real Income, Real Wage, and Population in England: 1200-1860

Data Source: Clark (2009).

6.1.1 Time, Generational Growth Rates, and the Initial Period

Throughout the analysis, the model's initial period is denoted by $t = 1$ and the length of a period is taken to be 25 years, i.e. the lifetime of one generation. The growth rates reported are hence *generational growth rates* per 25 years. The conversion to the annual rates is not necessary since the interest is limited with the long-run evolution of the economy.

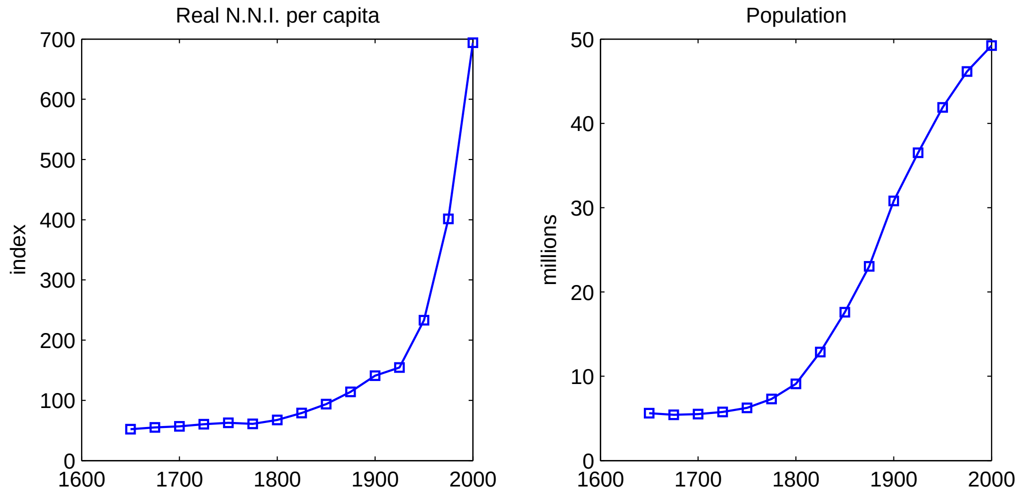

Figure 6.2: Real NNI per capita and Population: 1650-2000

Notes: The original data source for both series is Clark (2009). Since the raw data are decennial, mid-decade levels for 1675, 1725, and so on are calculated using simple arithmetic average between closest decennial data. That is, the level of real N.N.I. per capita in 1675, e.g., is equal to the average of those in 1670 and 1680 and so on.

The initial period $t = 1$ of the model economy is matched with the year 1650. The reasoning behind this choice is threefold:

First, as argued earlier, the model is not designed to capture the Malthusian cycle between the real wage and the level of population. Since this type of Malthusian dynamism in England ends at around the year 1650 after which both population and living standards exhibit exponential growth, matching $t = 1$ with the year 1650 is not restrictive for the present purposes (see Figure 6.1).

Second, the year 1650 is early enough to let the model economy to reveal the dynamics before the industrial revolution. Normalizing the start date of the latter to 1750, matching $t = 1$ with the year 1650 gives us four generations, i.e. a century, to analyze the growth of K_t and N_t before the industrial revolution.

Finally, satisfactorily rich data exist for England only for the period after mid-1500s. As we shall see next, this is true both for the demographic variables and for the labor and the output shares of the traditional sector.

Figure 6.3: The Decline of the Traditional Sector: 1650-1975

Notes: The original data for the labor share of the modern sector is collected by Clark (2001) for England, covering the period 1565-1865, and by Maddison (1995) for the United Kingdom, covering the period 1820-1992. The data on the output share of the modern sector is generated by Bar and Leukhina (2010a) using the raw data from Clark (2001), Clark (2002), and Mitchell (1975). See Appendix A.1 of Bar and Leukhina (2010a) for details.

6.1.2 The Data

Living Standards and Population

The quantitative analysis of the model, first of all, requires a measure of living standards. This measure is *real net national income per capita* (real N.N.I. p.c.) estimated by Clark (2009).[21] The raw decennial data is next used to construct the time series from 1650 to 2000 with 25 year intervals by taking arithmetic averages for 25th and 75th years of any century.

The population in England is another key macroeconomic aggregate, and the raw data series is again borrowed from Clark (2009). This time series decennially runs from 1200 to 1860. To complete this series, the United Kingdom's (decennial) census data for

21. Specifically, nominal net national income is deflated with the price index of net domestic output for the period 1200-1860, and the index of real income per person is linked with the earlier series for 1860 and beyond. See Tables 28 and 34 in Clark (2009) for more details.

England from 1871 to 2001 are used.[22] Then, the decennial series is utilized to construct the dataset of 1650-2000 period with 25 year intervals using the same principles applied to the income data.

The Decline of the Traditional Sector

The measures of the structural transformation are the labor and the output shares of the traditional sector. The declines of these ratios are associated respectively with urbanization and industrialization processes as in Bar and Leukhina's (2010a) study, and Bar and Leukhina's (2010a) dataset on the labor and the output shares of the modern sector is used to generate the associated shares.[23] Figure 6.3 pictures the decline of the traditional sector for the period 1650-1975.

Survival Probability and Adult Longevity

Figure 6.4 pictures the actual data and the fitted values for survival probability s_t and (normalized) adult longevity ℓ_t. The fitted values for the period 1650-2250 are needed for the long-run simulations and used instead of the actual data to design counter-factual experiments in a systematic manner. Before discussing how these fitted values are obtained, however, it is necessary to clarify the sources and the generation of the actual data.

The actual data series for the survival probability for ages 0-25 is based on the data used by Bar and Leukhina (2010a).[24] Bar and Leukhina's (2010a) finalized data for s_t runs from the mid-year 1612.5 to the mid-year 1987.5 in 25 year intervals. Setting the survival probability for the mid-year 2012.5 to 0.995, the data series for the period 1625-2000 is

22. Strictly speaking, the census year is matched with the decade it starts; population of the year 1870 in the completed series is the level of population recorded in 1871 census, and so on. Population data from the U.K. censuses are available online at `www.statistics.gov.uk/` website.

23. The dataset used by Bar and Leukhina (2010a) is obtained via personal communication with the authors.

24. The original data sources are Wrigley et al. (1997) and Human Mortality Database. The latter is accessible online at `www.mortality.org/` website.

Table 6.1: The Regression Results for the Exogenous Variables

The Logistic NLS Fit of s_t			The Logistic NLS Fit of ℓ_t		
	Estimate	95% Con. Int.		Estimate	95% Con. Int.
μ_{s0}	0.63	[0.62, 0.65]	$\mu_{\ell 0}$	0.42	(0.40, 0.44)
μ_{s1}	1.15	[0.82, 1.48]	$\mu_{\ell 1}$	0.37	(0.31, 0.44)
μ_{s2}	12.83	[12.54, 13.11]	$\mu_{\ell 2}$	15.75	(15.38, 16.13)
S.S.E.	0.00502		S.S.E.	0.00432	
R.M.S.E.	0.01967		R.M.S.E.	0.01823	
Adj. R^2	0.97820		Adj. R^2	0.98230	

generated using arithmetic averages.[25]

The data on adult longevity is constructed as follows: Using the period life expectancy data of Wrigley et al. (1997) for the periods before 1825 and of the United Kingdom National Statistics for the periods after 1850, the period life expectancy at age 25, denoted by e_{25}, is retrieved. Then, setting the maximum lifespan of humans to 100 years, the normalized adult longevity corresponding to ℓ_t in the model, is generated by dividing e_{25} to 75 for all periods.

The fitted values for s_t and ℓ_t are obtained by estimating the following simple logistic equations via Nonlinear Least Squares

$$s_t = \mu_{s0} + \frac{1-\mu_{s0}}{1+\exp\left[-\mu_{s1}(t-\mu_{s2})\right]} + \epsilon_{st}$$
$$\ell_t = \mu_{\ell 0} + \frac{1-\mu_{\ell 0}}{1+\exp\left[-\mu_{\ell 1}(t-\mu_{\ell 2})\right]} + \epsilon_{\ell t}$$

where t takes integer values starting from unity and ϵ_{st} and $\epsilon_{\ell t}$ are error terms. Table 6.1 summarizes the estimation results.

25. We shall see below that the value of s_t in 1625, i.e. in the model period $t=0$, is needed to obtain a model-based data counterpart for fertility.

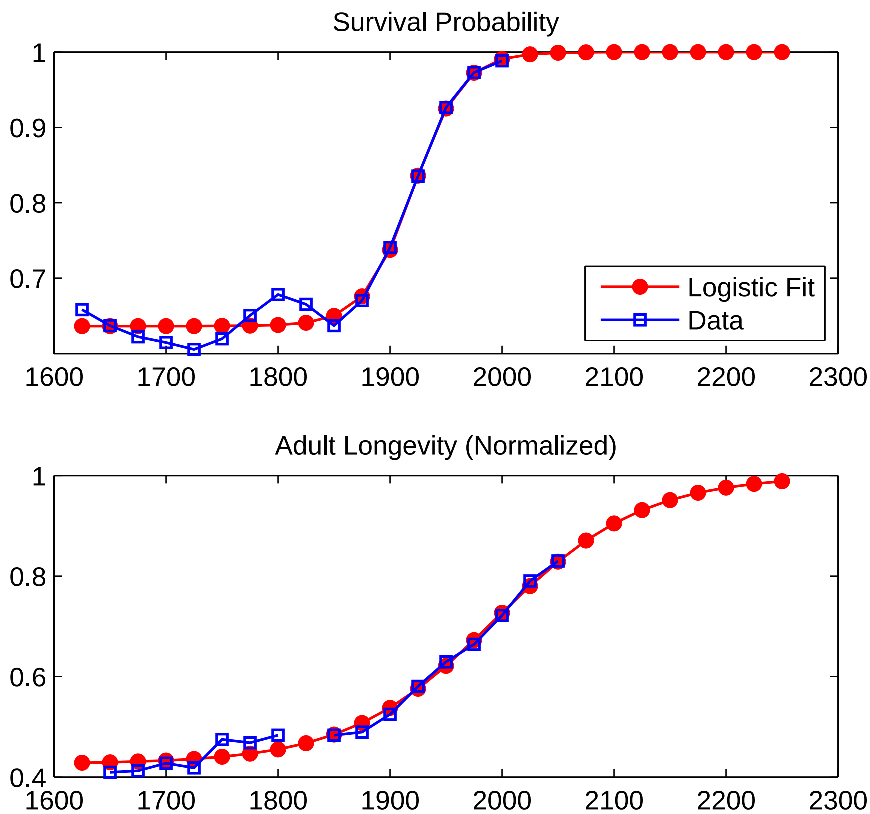

Figure 6.4: Survival Probability and Adult Longevity: Data vs. Fitted Values

Notes: See the text for a description of actual data. Fitted values are obtained through Nonlinear Least Squares estimation of the logistic functions introduced. See Table 6.1 for the regression results.

Fertility Data

A data series needed for the quantitative analysis is that of gross fertility. The first alternative here would be to use the actual data on total fertility rate since it is explicitly defined as a model variable. This however is not a completely fair way to judge the model's capability of explaining fertility and population dynamics since the demographic structure of the model is extremely simple with two generations, asexual reproduction, and a common gross fertility level for all adult individuals. Another alternative would be to derive the model-based counterpart of the crude birth rate or the general fertility rate while making some necessary corrections using the survival probabilities for different ages and

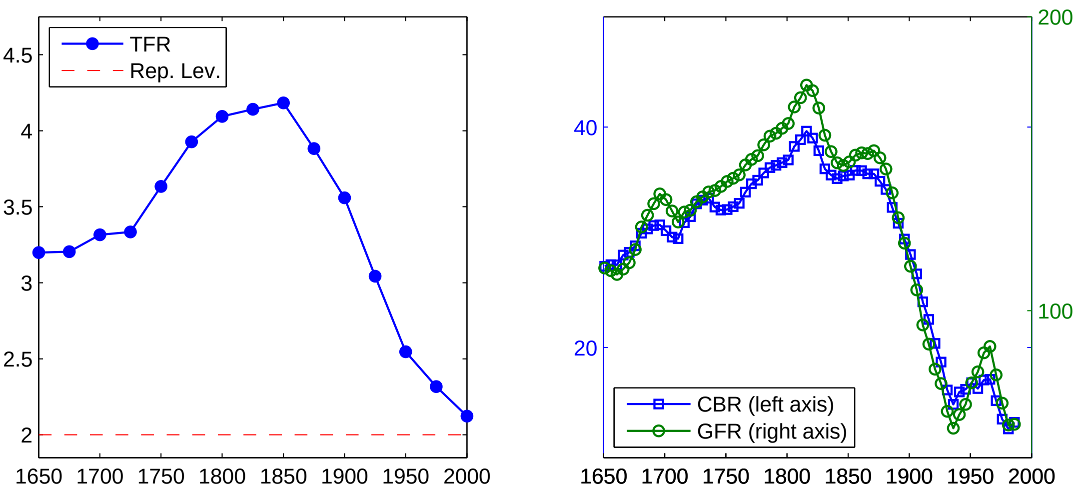

Figure 6.5: The Measures of Gross Fertility in England: 1650-2000

Notes: Crude birth rate (CBR) and general fertility rate (GFR) measure the annual number of live births respectively per 1000 people and per 1000 women of childbearing age. The dashed line indicates the replacement level for a survival probability of unity.

mid-year population levels. Yet, there is a third alternative which is clearly less tedious than the second one. Since we have actual data series of population and survival probability, the model-based counterpart of gross fertility per adult can be easily obtained using

$$b_t = \left(\frac{P_t}{P_{t-1}}\right)\left(\frac{1}{s_{t-1}}\right)\left(\frac{1+b_{t-1}}{b_{t-1}}\right) - 1$$

in a recursive way where this equation follows from $P_t = (1+b_t)N_t$ and $N_{t+1} = s_t b_t N_t$. The model-based data counterpart of total fertility rate is thus $2b_t$.

Figure 6.5 contrasts model-based total fertility rate described above with the actual data on general fertility and crude birth rates collected by Bar and Leukhina (2010a).[26] Clearly, and not surprisingly, the derived data, pictured in the upper-panel of Figure 6.5, closely represents the dynamics of gross fertility measured by general fertility and crude

26. To generate the model-based total fertility rate data, smoothed population series and the logistic fit of survival probability is used. $2b_{t-1}$ is taken to be 3.50. This is the maximum level of total fertility rate that implies a non-decreasing adult population after 1650. Moreover, the dependency of b_t series on this initial value dies out rather quickly after 2 to 3 periods.

birth rates. More importantly, if the model economy is successful in explaining the dynamics of model-based total fertility rate, it comes closer to explain the dynamics of the level of population since s_t is exogenous.

The Share of Entrepreneurs

The last piece of data to be introduced, actually a single data point, is of the share of entrepreneurs in adult population. The ratio $1 - e_t = (N_t - E_t)/N_t$ in the model is the share of all working-age individuals employed as (manual) workers. Thus, the share of individuals working as non-manual workers such as managers and business owners corresponds to the ratio e_t in the model. Fortunately, the data collected by Routh (1987) exactly measures this: In 1975, 26.075% of all working-age individuals in England were non-manual workers.

6.1.3 Parameterizing the $\xi(K)$ Function

The last task before proceeding to the discussion of how the model parameters and initial values are calibrated is to specify an explicit form for $\xi(K)$ function. Since there does not exist an empirically guided way of characterizing $\xi(K)$ function beyond the restrictions put earlier via (3.10), the quantitative analysis is based on the simplest functional form satisfying these restrictions:

$$\xi(K_t) = 1 - \frac{1}{1 + K_t}$$

The reason behind specifying $\xi(K_t)$ without introducing a new structural parameter is an identification difficulty. $\theta > 0$ that represents the extrinsic productivity of unit inventive effort is already separated from $\xi(K_t)$, and a parameter changing $\xi(K_t)$ for a given value of K_t would not be identified without some additional data on discovery-invention relationship.[27]

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27. Note that another very simple form of $\xi(K_t)$ that satisfies (3.10) is $\xi(K_t) = 1 - \exp(-K_t)$. Using this functional form however does not change the qualitative nature of results.

6.1.4 Calibration

To simulate the model via forward recursion, all model parameters and the initial values of all endogenous state variables for $t = 1$, i.e. $(N_1, K_1, \overline{X}_1, X_{T1})$, must be calibrated to unique values. The purpose of the calibration exercise is to let the actual data determine the calibrated values wherever possible.

Two difficulties with the calibration are the following: First, the model's asymptotic equilibrium, as an equilibrium at which most variables are fixed at their asymptotic levels, cannot be used for the calibration basically because actual data do not exist for $t \to \infty$.[28] This difficulty implies that a simulated method of moments type of calibration strategy must be used; an algorithm that solves the model for a given set of parameters and initial values and minimizes a quadratic form of deviations between model predictions and actual data. Second, since there exist a total of 12 equilibrium regimes and the purpose is to match the observed timing of the industrial revolution and the demographic transition, there exist *hidden constraints* on parameters and initial values. That is, the benchmark set of parameters and initial values must ensure that the economy is in Regime I at $t = 1$ and an industrial revolution starts at $t = t^{tr} = 5$ that corresponds to 1750.

The calibration strategy is mixed: First, the value of a parameter is borrowed from the literature to ease the calibration exercise. Second, each initial value is normalized or calibrated from the data. Third, some identified parameters are solved using the relevant data points that define a set of data targets. Finally, the remaining parameters are jointly calibrated. That is, extending the set of data targets, a numerical algorithm is used to determine the values of these parameters that minimize an objective function measuring a distance between model predictions and data targets.

The labor exponent $\eta \in (0, 1)$ of the traditional technology is set to 0.537 which is calibrated by Bar and Leukhina (2010a) for England; the traditional technology postulated

28. By featuring $n_t = 1$ and $f_t^{NM} = 0$, the model's asymptotic equilibrium is in fact not completely informative about all of the parameter values. Strictly speaking, though, the asymptotic growth rate of modern sector productivity is restricted to identify one of the parameters; see below.

by Bar and Leukhina (2010a) isolates the input of labor hours just as in this model.

The initial value N_1 of adult population, measured in millions, is solved from $N_1 = P_1/(1+b_1)$ where both population P_1 and gross fertility per adult b_1 are actual data values. The calibrated value of N_1 is 2.0718 millions.

K_1 is normalized to unity. Notice that K_1 and ω both affect the timing of the industrial revolution. The normalization thus identifies ω given (i) the period at which the industrial revolution starts, and (ii) other determinants of the growth of K_t.

Also normalized to unity is $\overline{X}_1$. The arbitrariness here does not matter quantitatively since (i) output (per capita) data is available as an indexed variable and (ii) there always exists a value of $\gamma > 0$ that implies a given level of initial fertility for any value of $\overline{X}_1$. When the model predictions and the actual data are contrasted below, the model predictions of output (per capita) and real wage, obtained under the normalization $\overline{X}_1 = 1$, are reindexed without any affect on the benchmark calibration results.

The last initial value X_{T1} is identified only if we know ρ and λ. It turns out that to calibrate all three of them in a single system of three equations is feasible. The first equation of this system is (4.6) for $t = 1$ which is rewritten here as

$$\frac{N_{T1}}{N_1} = \left(\frac{1}{\ell_1 N_1}\right)\left(\frac{X_{T1}}{(1-\lambda)^{1-\lambda}\lambda^{\lambda}\overline{X}_1^{\lambda}}\right)^{\frac{1}{1-\eta}} \tag{6.1}$$

where the labor share of the traditional sector in 1650 solves X_{T1} given λ since we already have $\left(\eta, N_1, \overline{X}_1\right)$ and ℓ_1 from the fitted data. The second equation of the system, again evaluated at $t = 1$, solves the output share of the traditional sector. Using the results derived earlier, we can rewrite this equation as

$$\frac{Y_{T1}}{Y_{T1}+Y_1} = \frac{\dfrac{X_{T1}^{\frac{1}{1-\eta}}}{\left((1-\lambda)^{1-\lambda}\lambda^{\lambda}\overline{X}_1^{\lambda}\right)^{\frac{\eta}{1-\eta}}}}{\dfrac{X_{T1}^{\frac{1}{1-\eta}}}{\left((1-\lambda)^{1-\lambda}\lambda^{\lambda}\overline{X}_1^{\lambda}\right)^{\frac{\eta}{1-\eta}}} + (1-\lambda)\left(1-\frac{N_{T1}}{N_1}-\frac{\rho b_1}{\ell_1}\right)N_1\left(\frac{\lambda}{1-\lambda}\right)^{\lambda}\overline{X}_1^{\lambda}\ell_0} \tag{6.2}$$

where b_1, again, is a data point. Finally, the third equation solves the ratio of entrepreneurs in adult population from (4.9) for the year 1975 which corresponds to the 14th period of the model:

$$\left(\frac{E_{14}}{N_{14}}\right) = (1-\lambda)\left(1 - \frac{N_{T14}}{N_{14}} - \frac{\rho b_{14}}{\ell_{14}}\right) \tag{6.3}$$

The set of (first-stage) data targets of the calibration exercise are thus

$$2b_1, 2b_{14}, \frac{N_{T1}}{N_1}, \frac{N_{T14}}{N_{14}}, \frac{Y_{T1}}{Y_{T1}+Y_1}, \frac{E_{14}}{N_{14}}$$

Using data values of these targets in advance, the unique solution to the system (6.1)-(6.3) is (numerically) solved such that $X_{T1} = 0.3951$, $\lambda = 0.7102$, and $\rho = 0.0419$.

Next, recalling that the initial gross growth rate ζ_0^{η} of the traditional sector productivity should approximately be equal to $n_1^{1-\eta}$ for output per worker to stagnate before the industrial revolution, ζ_0 is calibrated to 1.0154 given $n_1 = s_1 b_1$ from the data and $\eta = 0.537$.

The remaining parameters are θ and $\boldsymbol{\pi} \equiv (\sigma, \zeta_1, \omega, \phi, \gamma, \psi)$. The former is identified for any value of σ through the normalization of the asymptotic annual growth rate of productivity to 2.5% given λ. Such a restriction is necessary because the model must *not* be featuring a very large asymptotic growth rate that would be counter-factual to the observed acceleration of the growth rate in the 20th century.[29] Returning to the parameters defining the vector $\boldsymbol{\pi}$, a numerical algorithm is used to determine the values of these five parameters; see below. Clearly, the resulting value of σ returns the value of θ.

Let $\mathbf{m}^{data} \in \mathbb{R}_+^d$ and $\mathbf{m}(\boldsymbol{\pi}) \in \mathbb{R}_+^d$ respectively denote the vectors of actual data and

29. Recall that the model predicts that productivity growth accelerates towards the asymptotic equilibrium given increasing K_t and ℓ_t.

model predictions where $d \in \mathbb{N}_{++}$. The objective function to be minimized is defined as

$$Q(\boldsymbol{\pi}) \equiv \sum_{r=1}^{d} \left[\frac{m_r^{data} - m_r(\boldsymbol{\pi})}{0.5(m_r^{data} + m_r(\boldsymbol{\pi}))} \right]^2$$

where m_r^{data} and $m_r(\boldsymbol{\pi})$ correspond to r^{th} dimension of associated vectors, and $\mathbf{m}^{data} \in \mathbb{R}_+^d$ and $\mathbf{m}(\boldsymbol{\pi}) \in \mathbb{R}_+^d$ vectors are of dimension $d = 17$ such that the targets include

- log(Output per capita) and log(Population) in 1650, 1750, 1850 and 2000,
- the labor and the output shares of the traditional sector in 1650, 1750, 1850 and 1975, and
- the share of entrepreneurs in 1975.

1650 data targets are included because it is the initial period ($t = 1$), and 1750 ($t = 5$) is, again, the period at which industrial revolution starts. 1850 ($t = 9$) and 2000 ($t = 15$) are respectively the periods at which gross fertility is at its historical maximum and minimum; the data for these periods are informative for parameters (ϕ, γ, ψ). Finally, 1975 ($t = 14$) is included because the share of entrepreneurs and the labor and the output shares of the traditional sector are available for these periods.

The numerical algorithm that returns $\boldsymbol{\pi}$ minimizes $Q(\boldsymbol{\pi})$ under two hidden constraints. These constraints that ensure the SGE of period $t = 1$ is in Regime 1 are

- $K_1 < K_1^{tr}$ for the invention regime, and
- $W_1 < (\phi + \gamma)/\ell_1$ for the fertility regime.[30]

The sampling algorithm `imfil` developed by C. T. Kelley is used to handle these hidden constraints.[31] This algorithm searches the parameter space for a global optima

30. Since $a_1 = 0$, the spillover from the modern to the traditional sector is not active at $t = 1$. Thus, there does not exist a hidden constraint for this.

31. The code and the documentation are available online at `www4.ncsu.edu/~ctk/imfil.html` website.

Table 6.2: The Benchmark Calibration Results

Initial Values	Symbol	Value
Adult Population	N_1	2.0718
The Stock of Discoveries	K_1	1.0000
Mod. Sec. Productivity	$\overline{X}_1$	1.0000
Trad. Sec. Productivity	X_{T1}	0.3951

Parameters	Symbol	Value
Labor Exponent of the Trad. Tech.	η	0.5370
Constant Growth of Trad. Sec. Prod.	ζ_0	1.0154
Sectoral Spillover Parameter	ζ_1	1.1877
Labor Exponent of the Mod. Tech.	λ	0.7102
Collective Discovery Parameter	ω	1.3150
Gross Stepsize of Inventions	σ	1.4866
Extrinsic Prod. of Inventive Effort	θ	2.3952
"Subsistence" Consumption	γ	0.1367
Fertility Preference	ϕ	0.1223
Good Cost of Reproduction	ψ	0.0376
Time Cost of Reproduction	ρ	0.0419

while ignoring the parameter vectors that violate the hidden constraints. The algorithm has been modified by randomized initial iterates since the exercises have shown that, for some initial iterates, the algorithm returns a local optima of $Q(\boldsymbol{\pi})$. Specifically, 3 million initial iterates for $\boldsymbol{\pi}$ have been randomly generated, and around 1 million of these initial iterates have satisfied the hidden constraints of $K_1 < K_1^{tr}$ and $W_1 < (\phi + \gamma)/\ell_1$ and the imposed target $t^{tr} = 5$. The benchmark $\boldsymbol{\pi}$ is the one that implies the unique global minimum of the objective function among those obtained for these 1 million or so random iterates. The benchmark calibration results for all parameters and initial values are collected in Table 6.2.

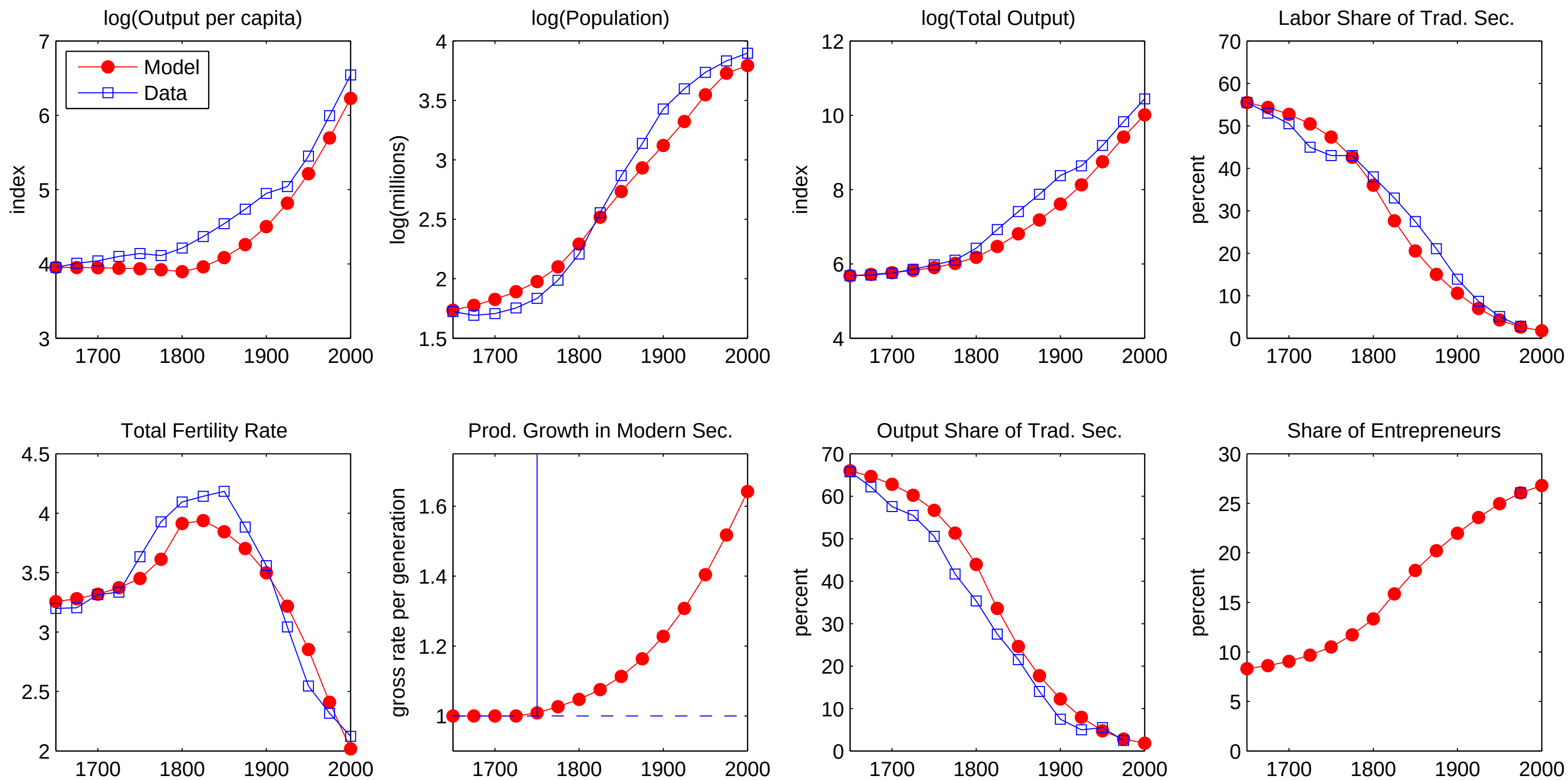

Figure 6.6: Model vs. Data

Notes: Filled circles and squares respectively indicate the model predictions and the actual data described earlier. In the figure that shows the productivity growth rate of the modern sector, the vertical and the horizontal lines respectively denote the period 1750 and the no-growth baseline. The model's predictions for level variables of output are reindexed without any effect on the benchmark calibration results.

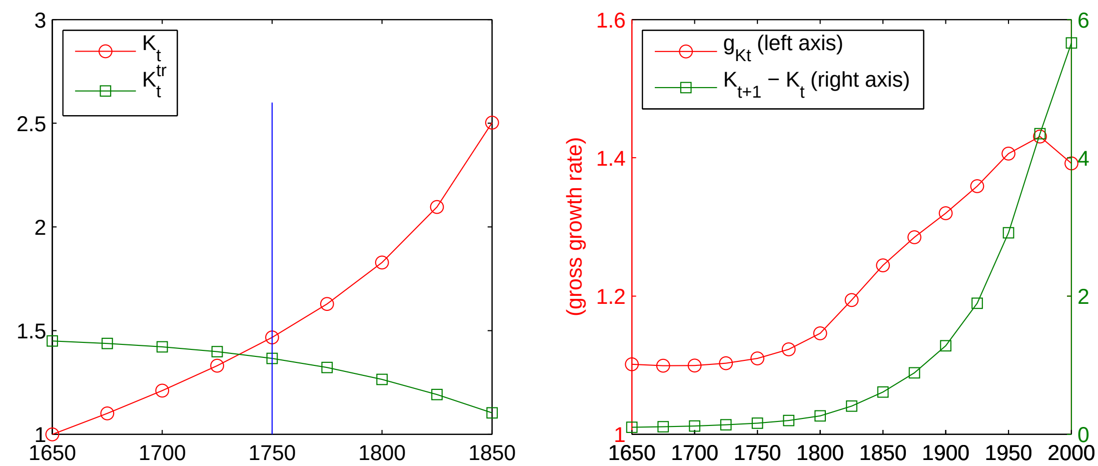

Figure 6.7: The Invention Threshold and the Collective Discovery

Notes: The vertical line on the left panel indicates the period 1750.

6.2 Results

6.2.1 Some Simulations for the Calibration Period 1650-2000

Of prime interest is how the model performs in predicting the observed patterns, and Figure 6.6 contrasts the model predictions with the actual data for the calibration period.

The model does not *perfectly* predict the actual data on output per capita, population, fertility, and so on. On the other hand, even for the variables regarding which the model performs least successfully, the overall pattern is captured by the benchmark calibration.

What requires a comment is the limited success of the model in predicting output *per capita* before mid-19th century. Since the level of population in the model is simply measured by $(1 + b_t)N_t$, the fast increase in total fertility rate $2b_t$ before 1825 leads the model to predict a *declining* level output per capita before 1800. On the other hand, total output is matched more successfully since it is not affected by how the level of population in the model is defined.

Figure 6.7 pictures the stock K_t of useful discoveries and discloses the dynamics of collective discovery. In the left panel, the decrease of the time-varying threshold K_t^{tr} with

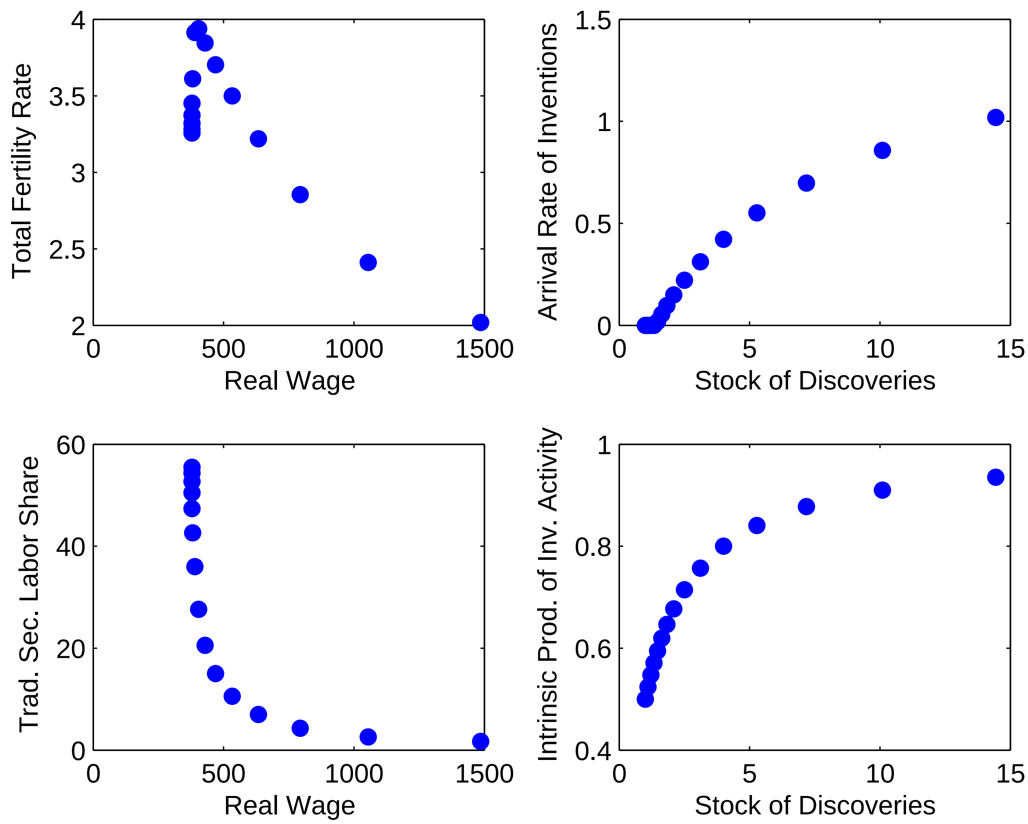

Figure 6.8: Some Equilibrium Relations

Notes: The model's predictions for real wage are reindexed without any affect on the benchmark calibration results.

increasing ℓ_t and the expansion of K_t are pictured. Naturally, the industrial revolution starts at 1750 at which $K_t > K_t^{tr}$. In the right panel, on the other hand, the gross growth rate g_{Kt} and the mass $K_{t+1} - K_t$ of new discoveries are shown. Increasing E_t and ℓ_t after 1800 explain the accelerating pace of collective discovery and the decelerating effect of stabilizing population is observed in the last period.

In Figure 6.8, some equilibrium relations are pictured for the period 1650-2000. The top left panel show the relationship between real wage and fertility, and the pull effect of real wage on the traditional sector is observed in the bottom left panel. In the top right panel, the arrival rate a_t of inventions is shown to increase with the stock of discoveries. The intrinsic productivity $\xi(K)$ of inventive activity, as it is presumed, is shown in the

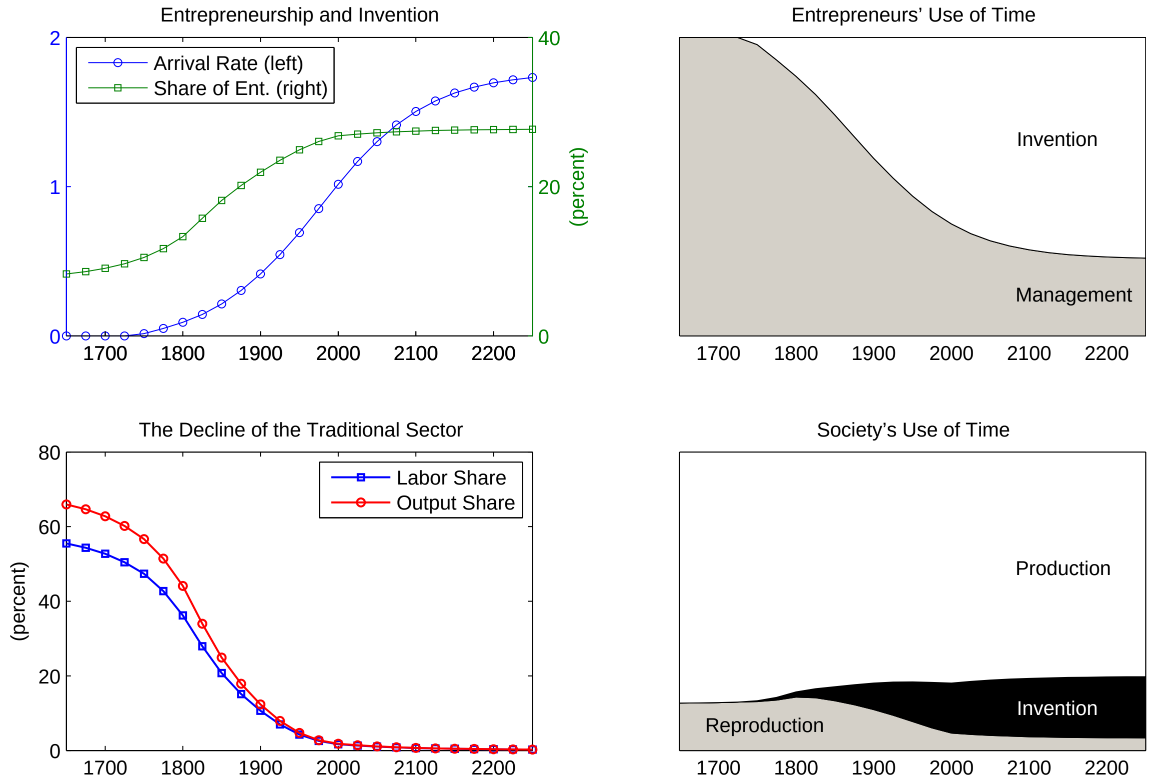

Figure 6.9: The Long-Run Evolution of the Equilibrium

Notes: In the right panels, the areas of the figures represent, for each t, the total available time endowments ℓ_t and $\ell_t N_t$.

bottom right panel.

6.2.2 The Equilibrium Path over the Long-Run

The long-run evolution of the model's static general equilibrium is pictured in Figure 6.9. The top left panel shows how the arrival rate a_t of inventions and the share e_t of entrepreneurs in adult population increase during the transition and converge to their asymptotic values. Importantly, the arrival rate exhibits a logistic shape because of slowly increasing (K_t, ℓ_t) in the early stages of the industrial revolution. The top right and the bottom right panels respectively show the entrepreneurs' and the society's optimal use of time where the areas of the figures represent, for each t, the total available time

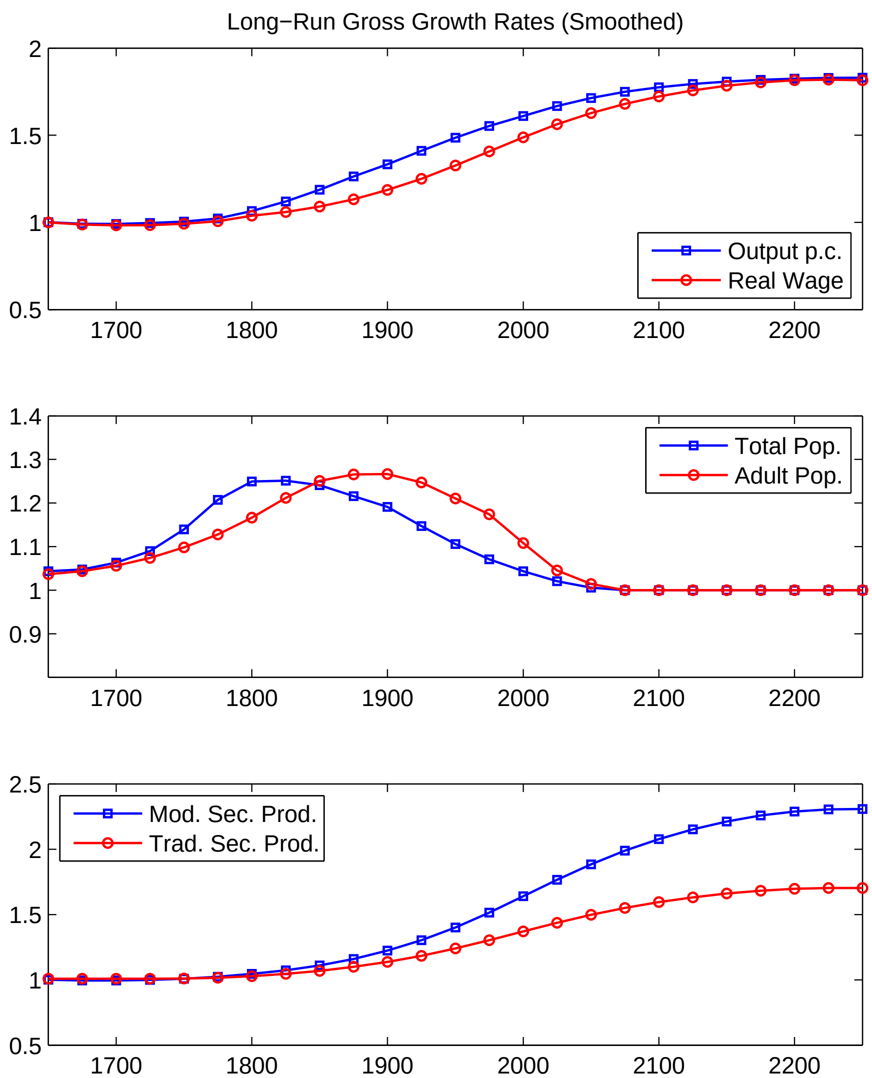

Figure 6.10: Long-Run Gross Growth Rates (Smoothed)

Notes: The long-run growth rate of each variable is smoothed via robust loess procedure with a span of 7 observations.

endowments ℓ_t and $\ell_t N_t$. The transition to the asymptotic equilibrium is characterized by the decline of management's share to its asymptotic value while invention's share, as its rival, increases. For the society, a large fraction of total available time endowment is spent on production (including management hours) with a slight decrease in time while the remarkable trade-off is predicted to occur between reproduction and invention. That is, the society from pre-industrial times to the modern era finds it optimal to invest more

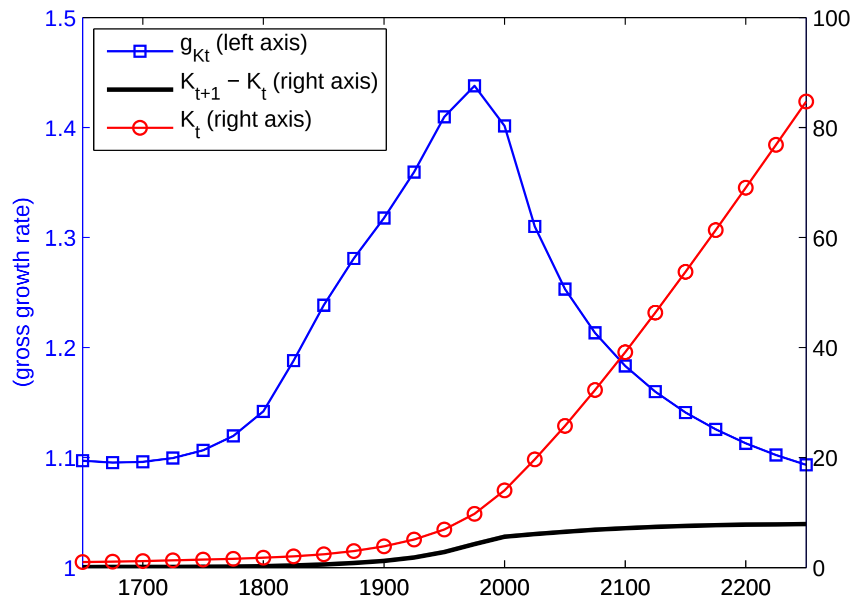

Figure 6.11: The Process of Collective Discovery over the Long-Run

in an *intangible* "asset", i.e. knowledge, and less in a *tangible* "asset", i.e. children.

Returning to the long-run growth rates, the model's predictions for long-run growth of living standards, population and productivity are shown in Figure 6.10. The smoothed growth rates indicate the model's overall success in explaining unified growth phenomena. Further, the lateness of the peak of adult population growth indicates population aging and the wedge between the sectoral productivity growth rates hints the decline of the traditional sector once again.

The last figure to be discussed in this subsection, Figure 6.11, pictures the long-run pattern of collective discovery. As argued earlier, the growth rate g_{Kt} converges to its asymptotic level of unity. While this is happening, the stock of K_t itself attains a linear time-series, converging to ∞, and the total lifetime $E_t \ell_t$ of entrepreneurs converges to the constant $E^\star \ell^\star$. Thus, collective discovery continues at an always decreasing rate. On the other hand, recall that the inventive activity is not affected by how big the stock of discoveries is *for* $t \to \infty$.

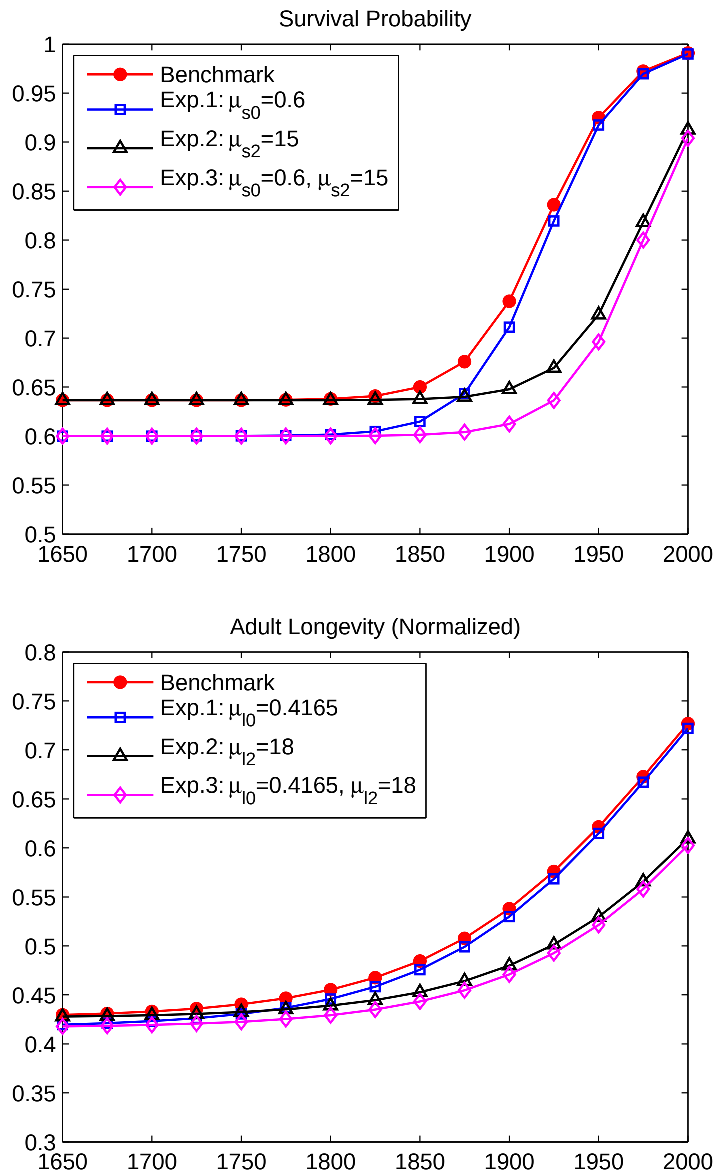

Figure 6.12: The Experimental Specifications of Exogenous Variables

Notes: See the text for a description of experimental specifications and the results.

6.2.3 Counter-factual Experiments and the Timing Question

Inspired by Desmet and Parente (2009), this subsection presents the results of some counter-factual experiments on the timing of the industrial revolution. These experiments are important because, once again, the analytical results presented in the previous chapter do not reveal the overall effects of model inputs on the timing of the industrial

Table 6.3: The Timing Effects of Model Parameters

Parameter	Symbol	Experiment	Industrial Revolution
Fertility Preference	ϕ	—	No Effect
"Subsistence" Consumption	γ	5% Higher	Delayed for 1 generation
Time Cost of Reproduction	ρ	25% Higher	Delayed for 2 generations
Good Cost of Reproduction	ψ	10% Higher	Delayed for 1 generation
Collective Discovery	ω	50% Lower	Delayed for 2 generations
Labor Exp. of Mod. Tech.	λ	10% Lower	Delayed for 6 generations
Stepsize of Inventions	σ	25% Lower	Delayed for 3 generations
Extrinsic Prod. of Inv. Effort	θ	50% Lower	Delayed for 6 generations
Con. Gr. of Trad. Sec. Prod.	ζ_0	5% Higher	Delayed for 1 generation
Sectoral Spillover Parameter	ζ_1	—	No Effect
Labor Exp. of Trad. Tech.	η	25% Lower	Delayed for 1 generation

revolution due to the second-order effects. Besides, building on the benchmark calibration results, the timing question can be approached specifically for the England's industrial revolution. The results, however, should still be read only as suggestive because of the simplicity of the model economy and the "ceteris paribus" assumption behind the experiments.

The first set of experiments investigates the role of differing survival probability (s_t) and adult longevity (ℓ_t) levels. For each exogenous variable, some of the estimated parameters of logistic fits are altered ceteris paribus. Specifically, for s_t, (i) μ_{s0} is decreased from 0.63 to 0.6 in Experiment 1, (ii) μ_{s2} is increased from 12.83 to 15 in Experiment 2, and (iii) Experiments 1 and 2 are merged to design Experiment 3. Similarly, for ℓ_t, $\mu_{\ell 0}$ is decreased by 0.01 units to 0.4165 and $\mu_{\ell 0}$ is increased to 18. Figure 6.12 shows the resulting experimental inputs.

The results indicate that even such small changes in s_t and ℓ_t create effects on the timing of the industrial revolution. For s_t, Experiments 1 and 3 imply that the industrial revolution is delayed by one generation, starting at 1775, but no hastening or delaying effect is found in Experiment 2. For ℓ_t, Experiment 1 and 2 also imply a one-generation delay, but the delay is equal to three generations when these changes merged in Experiment 3.

The second, the third and the fourth sets of experiments respectively investigate the effects of (i) $(\phi, \gamma, \rho, \psi)$ that directly affect population growth, (ii) $(\omega, \lambda, \sigma, \theta)$ of invention and collective discovery technologies and (iii) (ζ_0, ζ_1, η) that governs the productivity growth and the size of the traditional sector. Table 6.3 summarizes the results.

First recalling that fertility preference ϕ does not have threshold and growth effects when the industrial revolution occurs before the fertility decline, the timing of the industrial revolution is not affected by ϕ under the benchmark calibration. Next note that the sectoral spillover parameter ζ_1 does not create timing effects as expected. Another noteworthy result is that the hastening growth effects of parameters γ and ψ are dominated by the second-order effects on population growth. That is, when γ and ψ are higher than their benchmark levels, *ceteris paribus*, the associated decreases in the rate of population growth before the industrial revolution decreases the growth rate g_{Kt} of useful discoveries with N_t/K_t now being allowed to change. Finally, it should be noted that the ambiguity of the effect of λ is resolved such that the threshold effect dominates the supply of entrepreneurship effect. That is, when λ is lower than its benchmark level, there exist more entrepreneurs that collectively discover but invention becomes optimal later than the benchmark due to the smaller contribution of inventions to profits.

Overall, the counter-factual experiments show that small deviations from the benchmark model create large timing effects. Even if the model is not applied as a prototype model of the industrial revolution to economies other than England, the counter-factual experiments suggest that gains in adult longevity or the quality of collective discovery, e.g., might have played non-trivial roles in the timing of England's transition to modern economic growth. The bigger questions such as whether the first industrial revolution did not occur in China *really* because of the low quality of collective discovery there or whether the first industrial revolution could not have occurred in Sub-Saharan Africa *really* because of its disease environment remain open.

Chapter 7

Discussion

This chapter provides brief discussions on some aspects of the model and shows that the model can be extended in ways that may provide further insights on the role of entrepreneurship and knowledge for the Industrial Revolution.

7.1 The Industrial Revolution: Break or Continuity?

The unified growth model studied above has a stark implication: Entrepreneurs of a "special" generation find it optimal to direct resources into risky inventive activities unlike those of past generations. These entrepreneurs are "special" because the number of useful discoveries they have access to, given their longevity, is large enough to signal a higher expected level of profit for them if they are to decrease the time they could spend to routine management. In a sense, they benefit from standing on the shoulders of dead entrepreneurs who collectively created all these useful discoveries in a serendipitous way.

The invention threshold in the model leads to a kinked time-series of labor productivity in the modern production sector, and this in turn implies a kinked time-series of the real wage that exhibits exponential growth starting with the industrial revolution. The Industrial Revolution in history is matched by an invention revolution in the model. After this invention revolution, exerting inventive effort to appropriate an increasing profit remains optimal throughout the history. Figure 7.1 shows the data collected by Sullivan (1989) on the number of process innovations patented in England.

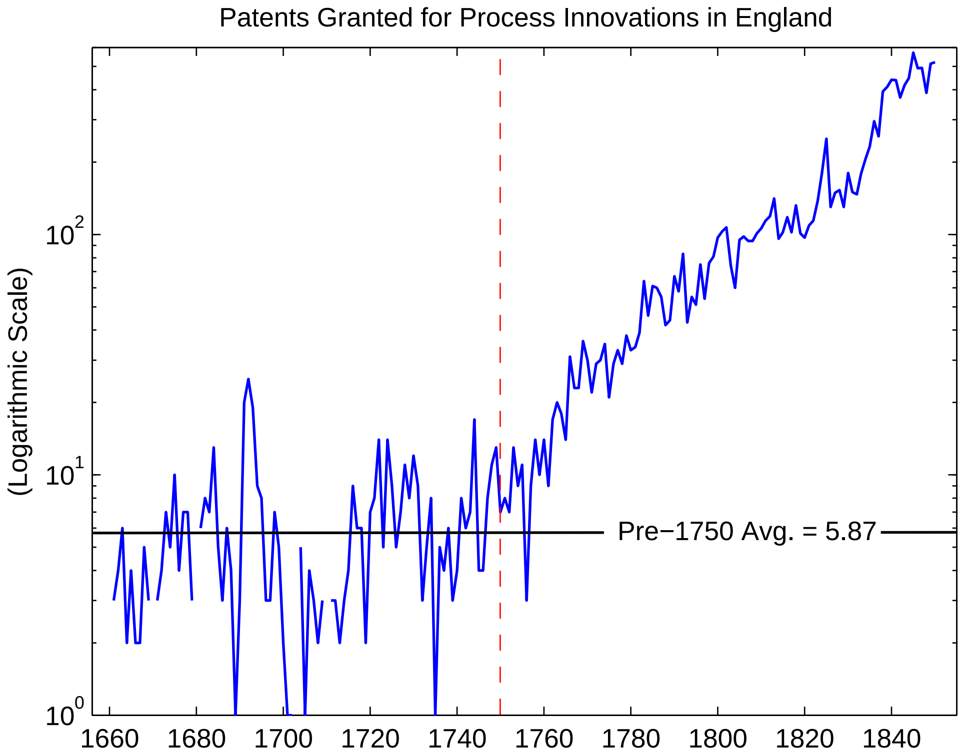

Figure 7.1: Patented Process Innovations in England, 1661-1850

Whether the first Industrial Revolution, roughly covering the period from 1760 to 1830, is a break from the past or a continuity has remained controversial among some economic historians. The *gradualist* view of Crafts and Harley (1992) suggests that there was little economic growth in England until the early 19th century in per capita terms and that the scope of fast technological progress was limited with the textile sector before the diffusion of the steam technology. The first argument has later been advanced by new estimates of Clark (2001), yet the notion of an industrial revolution as a structural break characterized by very slow growth in per capita terms is not controversial at all.[32] Studied extensively by Pereira (2003), several variables of interest, including total industrial output and population, exhibit endogenously determined upward breaks during the first Industrial Revolution, and Mokyr (2004) and others suggest that what kept output

32. As noted by Crafts's (2005, p. 533), "[g]radualism in the transition to modern economic growth should not be confused with an absence of fundamental change."

per capita at a very low level during the first Industrial Revolution was indeed the fast expansion of English population. [33]

The model economy constructed and studied above, as a unified model, captures exactly this type of dynamics between population and technology. The predicted time-series of output per capita does not exhibit growth for a couple of generations after the start of the industrial revolution, and increasing output due to endogenous technological progress hardly overcomes the population pressure until the pace of technological progress becomes fast enough.

The model economy's industrial revolution thus fits well with the notion of a structural break without conflicting with the view that a sort of continuity with the past exists. In essence, the period at which the industrial revolution in the model starts is determined by the latent dynamics of the model economy before the industrial revolution.

7.2 Serendipitous Inventions

That the rate of technological progress in the modern production sector *before* the Industrial Revolution is zero is counter-factual to what we observe in the data: As noted earlier, the real wage series in England has an upward trend after mid-1600s, and a minuscule rate of growth in the real wage before the Industrial Revolution is also consistent with the patent data of Sullivan (1989): The number of patented process innovations per year, albeit being trendless, implies minor improvements in productivity and hence in the real wage. A question of interest is thus whether the model can be extended to account for such haphazard type of technological progress.

The simplest extension along this line of thought is to allow for *serendipitous inventions* to exogenously increase the baseline productivity $\overline{X}_t$ of the modern sector.

33. Temin (1997), attacking the second argument of the gradualist view, shows that England was a net exporter not only in cotton textiles and iron goods sectors where technological progress was fast but also in many other industries. Crafts and Harley (2000), in defense, use a computable general equilibrium model that shows exports may increase in the absence of sectoral technological progress. Pereira (2003) rightly argues that those of Temin (1997) and Crafts and Harley (2000) were indirect tests.

Serendipitous inventions can be thought of as resulting exogenously without altering the optimal behavior of entrepreneurs regarding the inventive activity. The law of motion for $\overline{X}_t$ can simply be extended to include serendipitous inventions as in

$$\overline{X}_{t+1} = \overline{X}_t \exp[(\sigma - 1)(a_t + a_s)]$$

where $a_s > 0$ represents the arrival rate of serendipitous inventions under the additional assumption that a serendipitous invention has the same stepsize $\sigma > 1$ of a purposeful invention. Clearly, whenever $a_t = 0$, the gross growth rate of $\overline{X}_t$ reduces into $\exp[(\sigma - 1)a_s]$.

If allowed to be a positive number, the arrival rate a_s of technological progress through serendipitous inventions would affect the timing and the pace of demographic transition and structural transformation through the real wage channel. The qualitative nature of the model's analytical results, most notably the existence of an invention threshold, nevertheless remain same under this simple extension.

7.3 Managers vs. Inventors

One legitimate concern might be over the presumption that entrepreneur and inventor is the very same individual in the model. This presumption, recalling the motivating evidence by Meisenzahl and Mokyr's (forthcoming), is the simplest way to let inventors be incentivized via profit motive within the occupational choice framework adapted.

In a simple alternative formulation in which inventors are still incentivized by profit shares, entrepreneurs spend their entire labor endowment to management and make contracts with "freelance" inventors for the latter to undertake the inventive activities. Under certain simplifying assumptions, the invention threshold property of the basic model is preserved.

A "freelance" inventor is in essence a worker who may find it optimal to spend some of her labor endowment to inventive activities. To completely assume away search and matching frictions previously emphasized by Michelacci (2003), suppose that an inventor is always matched with an entrepreneur. To simplify the matters even more, let the contract between entrepreneur i and inventor i be such that an exogenously given fraction $1-\nu_t \in (0,1)$ of firm i's ex post profit is appropriated by inventor i.[34] This variable can thus be argued to represent a dimension of the quality of innovation-promoting institutions in period t as in Jones (2001).

Inventor i's expected (lifetime) earning under these assumptions can be written as

$$(1-\nu_t)\exp\left(\Sigma a_{it}\right)\Lambda\left(\frac{\overline{X}_t}{W_t}\right)^{\Gamma}\ell_t + W_t\left(\ell_t - \frac{a_{it}}{\theta\xi\left(K_t\right)}\right)$$

where the first term indicates the expected profit obtained by inventor i and the second term is her wage income. Since fertility choice by inventor i is still separable from the choice of a_{it}, optimal inventive effort is zero if

$$(1-\nu_t)\xi\left(K_t\right)\ell_t < \left[\theta\left(\sigma^{\frac{\lambda}{1-\lambda}}-1\right)\right]^{-1}$$

and the start of an industrial revolution not only requires a large enough stock of discoveries and a high enough adult longevity but also depends crucially on whether the society sufficiently rewards its potential inventors.[35]

7.4 The Fishing Out Effect

One noteworthy aspect of productivity growth in the simple version of the model is the absence of the so-called fishing out effect: The (unit) productivity of labor directed to inventive activity does not change and, hence, not decrease with the level of baseline

34. Ideally, ν_t could be allowed to follow endogenously from a bargaining problem between the entrepreneur and the inventor as in Michelacci (2003). A closed-form solution to the model in this case, however, does not exist.

35. Note that the unique SGE of this version of the model still features occupational choice through $\nu_t \mathrm{E}\Pi_{it} = W_t\ell_t = (1-\nu_t)\exp\left(\Sigma a_{it}\right)\Lambda\left(\frac{\overline{X}_t}{W_t}\right)^{\Gamma}\ell_t + W_t\left(\ell_t - \frac{a_{it}}{\theta\xi\left(K_t\right)}\right)$

productivity $\overline{X}_t$. No matter how high $\overline{X}_t$ is, the arrival rate a_{it} of inventions is fixed for given levels of K_t and H_{rit}. The endless expansion of K_t makes inventive effort always more productive in time as postulated, but the only limit realistically imposed is on the *level* of this epistemic effect when $K_t \to \infty$.

To incorporate the fishing out effect by making the arrival rate a_{it} a decreasing function of $\overline{X}_t$, the productivity term $\xi(K_t)$ of the model can be redefined as $\xi(K_t, \overline{X}_t)$ which satisfies

$$\frac{\partial \xi(K_t, \overline{X}_t)}{\partial \overline{X}_t} < 0 \quad \text{for all } K_t$$

Here, it is again required that, for any $\overline{X}_t < \infty$,

$$\frac{\partial \xi(K_t, \overline{X}_t)}{\partial K_t} > 0 \text{ for all } K_t < \infty \qquad \xi\left(0, \overline{X}_t\right) = 0 \qquad \lim_{K_t \to \infty} \xi\left(K_t, \overline{X}_t\right) = 1$$

Not surprisingly, the invention threshold and the asymptotic equilibrium of the model would be affected from the presence of the fishing out effect: A higher initial level $\overline{X}_0$ of modern sector productivity unambiguously delays the start of the industrial revolution. This, in a sense, is a technological lock-in result and is itself interesting as a theoretical possibility, but the hypothesis of induced innovation, applied to the Industrial Revolution by Allen (2011), *weakly* rejects this possibility to be true: The real wage in urban areas of Britain was higher in comparison to those of other European cities.

Returning to the asymptotic equilibrium, the fishing out effect may lead the asymptotic growth rate of modern sector productivity and output per capita to converge to zero. To be specific, let $\xi(K_t, \overline{X}_t)$ be defined as

$$\xi(K_t, \overline{X}_t) \equiv 1 - \frac{1}{1 + \frac{K_t}{\overline{X}_t}}$$

which is possibly the simplest functional form that satisfies the restrictions stated above. The advanced stages of economic development under this scenario are characterized by the decline of the knowledge-productivity ratio

$$k_t \equiv \frac{K_t}{\overline{X}_t}$$

to its invention threshold

$$k_t^{tr} \equiv \frac{1}{\theta\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)\ell_t - 1}$$

because the (gross) growth rate g_{Kt} of the stock of discoveries again converges to unity with the maximum level $N^\star$ of adult population being constant.[36] The model therefore asymptotically behaves like the semi-endogenous (unified) growth model of Jones (2001) and the model of Strulik and Weisdorf (2008) due to the fishing out effect. The actual very long-run growth rate of income per capita on the other hand does not exhibit a downward trend for the developed economies. Overall, the implications of the fishing out effect seem to be contradicting with observed patterns.

7.5 The Scale Effect in the Process of Collective Discovery

Since the mass E_t of entrepreneurs is proportional to adult population N_t, the process of collective discovery is characterized by a scale effect. Specifically, given the share e_t of entrepreneurs in adult population, the growth rate g_{Kt} of the stock of useful discoveries is increasing in N_t:

$$g_{Kt} = 1 + \frac{\omega \ell_t E_t}{K_t} = 1 + \frac{\omega \ell_t (1-\lambda)\left(1 - \frac{N_{Tt}}{N_t} - \frac{\rho b_t}{\ell_t}\right) N_t}{K_t}$$

This scale effect may raise the question of why economies that had bigger populations in pre-industrial era compared to England, e.g., China, did not achieve the first industrial revolution even though, as argued in Chapter 5, such questions should be formulated with great care. If taken seriously, still, an answer to the question is provided by the

36. Note that the asymptotic equilibrium thus characterized is globally stable.

determinants of collective discovery other than N_t: It is not solely the mere totality of adult hours $\ell_t N_t$ that explains the growth rate of K_t but also (i) the quality ω of the process of collective discovery and (ii) the share e_t of entrepreneurs in adult population.

As noted earlier, ω as a structural parameter represents the quality of the environment in which entrepreneurs create and share useful discoveries. England here had the advantage of being a small country with respect to (absolute) geographical size. Also advantages of England, as argued by those stressing the role of collective discovery and industrial enlightenment, are (i) the gentlemanly behavior and the technological motivation of business owners and (ii) the efficiency of social networks and informal institutions. A sufficiently large ω for England may well have dominated the negative effect of its comparatively small population.

Returning to the share of entrepreneurs, first note that pre-industrial fertility levels around the world were not significantly different. If one further assumes that pre-industrial longevity levels and parameters λ and ρ were similar in England and elsewhere, the prime determinant of the share of entrepreneurs would be the size of the traditional sector. The limited data here indicates that England in pre-industrial times had a higher rate of urbanization than China; see Voigtländer and Voth (2006).

In general, any rival use of time endowment is important in determining the supply or, put more correctly, the *lack* of entrepreneurship, and the labor shares of occupations that do not contribute to collective discovery would have delaying growth effects for the timing of the industrial revolution.

One such occupation regarding which England had arguably an advantage compared to China is state bureaucracy. Imagine, first, a richer framework with a political authority where bureaucrats are employed by the state to produce a public good. A larger state bureaucracy in this case would decrease the growth rate of K_t, *ceteris paribus*, because the mass of entrepreneurs who collectively discover would be smaller. This is true (i) whether the bureaucracy is financed through distortionary taxation or not and (ii) whether the productivity of a unit hour of a bureaucrat in producing the public good in question is increasing or stagnant. Obviously, if the bureaucrats work with stagnant productivity, sustained growth in social welfare is not possible before an industrial revolution, and a small state bureaucracy is a desired feature to increase the supply of entrepreneurship *ceteris paribus*. England might indeed have benefited from avoiding a large professional bureaucracy, as noted by Mokyr (1998), and China's potential might indeed have been restricted by its large and ineffective bureaucracy, as emphasized by Landes (2006).

7.6 Science and Scientists

The anecdotal evidence show that professional scientists' direct contribution to the *first* Industrial Revolution was limited. Also well-known is that British inventors, compared to those of other European nations, were particularly successful in applied sciences that built heavily upon the abstract contributions of, e.g., German and French scientists. That there does not exist a strong causality running from scientific progress to an industrial revolution is also supported by the fact that neither China nor the Islamic civilization, both scientifically superior to Europe at certain eras of antiquity, did realize an earlier industrial revolution. All these, together with the lack of reliable data on the number of scientists and a useful theoretical framework of the economics of science, motivate the model to exclude the role of science and scientists for the process of collective discovery.

A unified growth model that exploits the nexus between discoveries and inventions might nevertheless be expected to incorporate the role of science and scientists. One of the most important actor of the story of technological progress after the first Industrial Revolution is surely the professional scientist, and three questions, at least, remain open for the unified growth theory. First, why and how the grant-like forms of science patronage dominated the prize-like forms of it starting with the 18th century, a pattern documented by Hanson (1998), is central to the rise of professional scientists. Second, as emphasized by Pumfrey and Dawbarn (2004), science patronage exhibited a historical transition from being mostly ostentatious to being mostly utilitarian, starting first in the

16th century England. Finally, there does not exist a formal economic theory of the Scientific Revolution which was another major discontinuity in the history of mankind. A network model of the Scientific Revolution, once hinted by Kelly (2005), may be a useful starting point to understand the formation and the movement of European scientist networks between 1400 and 1900 documented by Taylor et al. (2008).

7.7 Endogenous Survival and Longevity

The model restricts the knowledge content of useful discoveries with the knowledge of natural phenomena that are related solely with the production processes. In a richer framework with scientists, discoveries could be thought of being applicable also to medicine. This second role of useful knowledge is exactly what Easterlin (1995) suggests; the Mortality Revolution of Europe did crucially depend on the creation and the diffusion of useful medical knowledge just as the Industrial Revolution benefited from the advances in physics, chemistry, geology, and other fields. In England, for example, the number of published books on health grew 9-fold from 1600 to 1800 as noted by de la Croix and Sommacal (2009).

In addition to the models that postulate a knowledge-mortality link via human capital accumulation, e.g. Cervellati and Sunde (2005) and Lagerlöf (2003), the conjecture has been formalized by Mokyr (1993) and more recently within unified growth frameworks by de la Croix and Sommacal (2009) and Strulik and Weisdorf (2011). Note that endogenizing survival probability (s_t) and adult longevity (ℓ_t) in a richer framework with scientists would pose no serious analytical difficulties. Given that (i) the stock of useful discoveries would still be growing along the transition and (ii) s_t and ℓ_t would still have the logistic shapes fitted against time in the present formulation, the only complication is to calibrate the parameters of two logistic functions of K_t, one for s_t and the other for ℓ_t.

Chapter 8

Concluding Remarks

A transition characterizes the economic history of today's developed economies. This is a transition from stagnating to growing levels of material living standards. It involves the declines of mortality and fertility rates with a bell-shaped pattern of population growth through time, i.e. the demographic transition. (Adult) longevity, the participation to and the length of formal education, the openness to trade, and the levels of urbanization and industrialization all increase along this transition. Democracy, the last but not the least, becomes persistent in the developed world where the transition had first started. In short, this is the transition to what social scientists call *modernity*. This, in Clark's (2007, p. ix) words, is *big history*.

The turning point of big history was the Industrial Revolution which represents a structural break in the sense that technological progress was no longer simply due to serendipitous inventions. Schumpeter's (1934) "entrepreneur-inventor"s, seeking increased market shares and profits, took the stage instead, and the world was not the same when the first corporate R & D lab was opened by Thomas Edison in 1876.

This dissertation studies a view of the Industrial Revolution that promotes the dual role of entrepreneurship for inventions and discoveries; the serendipitous expansion of the latter eventually leads to purposeful activation of the former. No such thing as an industrial revolution occurred for a very long episode of history because not enough was known about natural phenomena and lives were very short. Yet the type of useful knowledge relevant to production processes was created by and diffused among entrepreneurs. In one sense, it had to be because they were managing the firms utilizing these production processes.

The simple unified growth model of this mechanism constructed in this dissertation leaves many questions, other than the ones discussed in the previous chapter, open for further research. First, the emphasis is biased on the supply-side determinants of inventive activity, and the roles of market size and demand remain implicit within the two-occupation general equilibrium framework. How inventions are brought to markets as innovations is no less an important question. Second, the model simply features perfectly competitive innovation, and the role of patents for the industrial revolution, still controversial among economic historians, is ideally to be incorporated within a richer treatment. Third, the effects of knowledge diffusion on European and global scale, through the mobility of goods and people, is yet to be explored in a unified growth framework. The simple model of this dissertation may serve as a starting point of such an exploration. The last but not the least, how the rises of democracy and formal education are interrelated with *the enlightenment of the economy through useful knowledge* is an open question.

Appendix A

Proofs

Proof of Lemma 1:

(3.17′) can simply be rewritten as

$$\mathbf{E}\Pi_{it} = \exp\left(-a_{it}\right)(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{\overline{X}_t}{W_t}\right)^{\frac{\lambda}{1-\lambda}}\left(\ell_t - \frac{a_{it}}{\theta\xi\left(K_t\right)}\right)\sum_{z=0}^{\infty}\frac{a_{it}^{z}\sigma^{\left(\frac{\lambda}{1-\lambda}\right)z}}{z!}$$

after some arrangements. By Taylor's Theorem, the summation term on the right is identical to $\exp\left(\sigma^{\frac{\lambda}{1-\lambda}}a_{it}\right)$. Thus, we have

$$\mathbf{E}\Pi_{it} = \exp\left(\sigma^{\frac{\lambda}{1-\lambda}}a_{it} - a_{it}\right)(1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{\overline{X}_t}{W_t}\right)^{\frac{\lambda}{1-\lambda}}\left(\ell_t - \frac{a_{it}}{\theta\xi\left(K_t\right)}\right)$$

Defining and substituting $\Gamma \equiv \frac{\lambda}{1-\lambda}$, $\Lambda \equiv (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}$, and $\Sigma \equiv \sigma^{\frac{\lambda}{1-\lambda}} - 1$ yield (3.17″) and complete the proof. ■

Proof of Proposition 1:

The uniquely existing SGE follows from three features of the model:

1. Both decision problems have unique solutions since the objective functions are strictly quasi-concave and differentiably continuous on compact choice sets.

2. At these unique solutions, a unique level of real wage makes individuals indifferent between becoming an entrepreneur and becoming a worker.

3. This unique level of real wage is also the one that clears the labor market and, hence, that (residually) determines the mass of entrepreneurs.

Starting with the workers' problem characterized by (3.23) and (3.24), the Kuhn-Tucker F.O.C.s imply

$$\frac{\partial U_{wt}}{\partial n_{wt}} = -\left(\frac{1}{s_t}\right)(\rho W_t + \psi) + \frac{\phi}{n_{wt}} \begin{cases} < 0 & \text{if } n_{wt} = 1 \\ > 0 & \text{if } n_{wt} = \frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi} \\ = 0 & \text{otherwise} \end{cases}$$

Thus, optimal net fertility by workers satisfies

$$n_{wt} = \begin{cases} \frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi} & \text{if } W_t < \frac{\phi + \gamma}{\ell_t} \\ \frac{\phi s_t}{\rho W_t + \psi} & \text{if } W_t \in \left[\frac{\phi+\gamma}{\ell_t}, \frac{\phi s_t - \psi}{\rho}\right] \\ 1 & \text{if } W_t > \frac{\phi s_t - \psi}{\rho} \end{cases} \tag{A.1}$$

Next note that the entrepreneurs' problem characterized by (3.26)-(3.28) is separable in n_{it} and a_{it}. With respect to the latter, we have

$$\frac{\partial \mathbf{E} U_{it}}{\partial a_{it}} = \exp(\Sigma a_{it}) \Lambda \left(\frac{\overline{X}_t}{W_t}\right)^{\Gamma} \left[-\frac{1}{\theta \xi (K_t)} + \Sigma\left(\ell_t - \frac{a_{it}}{\theta \xi (K_t)}\right)\right] \begin{cases} < 0 & \text{if } a_{it} = 0 \\ = 0 & \text{if } a_{it} \in \left(0, a_t^{\max}\right) \end{cases} \tag{A.2}$$

Hence, there always exists a unique solution $a_{it} \geq 0$.[37] Moreover, this solution is symmetric, i.e. $a_{it} = a_t \geq 0$ for all i, since all entrepreneurs face the same set of given variables. In turn, the expected profit $\mathbf{E}\Pi_{it}$ is unique as well:

$$\mathbf{E}\Pi_{it} = \exp(\Sigma a_t) \Lambda \left(\overline{X}_t / W_t\right)^{\Gamma} \left(\ell_t - a_t / \theta \xi (K_t)\right) \tag{A.3}$$

37. Notice that (i) the S.O.C. for maximum is satisfied at this solution, and (ii) $a_{it} = a_t^{\max}$ is never optimal because it implies zero profits.

With respect to fertility n_{it}, the Kuhn-Tucker F.O.C.s imply

$$\frac{\partial \mathbf{E}U_{it}}{\partial n_{it}} = -\left(\frac{1}{s_t}\right)(\rho W_t + \psi) + \frac{\phi}{n_{it}} \begin{cases} < 0 & \text{if } n_{it} = 1 \\ > 0 & \text{if } n_{it} = \frac{(\mathbf{E}\Pi_{it} - \gamma)s_t}{\rho W_t + \psi} \\ = 0 & \text{otherwise} \end{cases}$$

Therefore, optimal net fertility by entrepreneurs satisfies

$$n_{it} = \begin{cases} \frac{(\mathbf{E}\Pi_{it} - \gamma)s_t}{\rho W_t + \psi} & \text{if } \mathbf{E}\Pi_{it} < \phi + \gamma \\ \frac{\phi s_t}{\rho W_t + \psi} & \text{if } \mathbf{E}\Pi_{it} \geq \phi + \gamma \text{ and if } W_t \leq \frac{\phi s_t - \psi}{\rho} \\ 1 & \text{if } W_t > \frac{\phi s_t - \psi}{\rho} \end{cases} \tag{A.4}$$

Lemma A.1: In the unique SGE, we have $n_{wt} = n_{it} = n_t$.

Proof— The intuition behind this lemma is twofold: First of all, the total cost of reproduction in terms of gross fertility, denoted by $\rho W_t + \psi$, is identical to workers and entrepreneurs. Second, entrepreneurs and workers, in equilibrium, are forced to derive equal (expected) utilities, i.e. $\mathbf{E}U_{it} = U_{wt}$.

Formally, the equal utilities restriction can be rewritten as

$$\mathbf{E}\Pi_{it} - (\psi + \rho W_t)\left(\frac{n_{it}}{s_t}\right) + \phi \ln(n_{it}) = W_t \ell_t - (\psi + \rho W_t)\left(\frac{n_{wt}}{s_t}\right) + \phi \ln(n_{wt}) \tag{A.5}$$

In what follows, n_{wt} is taken as a benchmark, and it is shown, for each characterization of n_{wt}, that $n_{wt} \neq n_{it}$ leads to a contradiction.

- The Case of $n_{wt} = \frac{(W_t \ell_t - \gamma)s_t}{\rho W_t + \psi}$:

  In this case, we must have from (A.1) that

$$W_t < \frac{\phi + \gamma}{\ell_t} \tag{A.6}$$

  - $n_{it} = 1$ in this case leads to a contradiction because (A.4) implies $W_t > \frac{\phi s_t - \psi}{\rho}$.

- $n_{it} = \frac{\phi s_t}{\rho W_t + \psi}$ requires $\mathbf{E}\Pi_{it} \geq \phi + \gamma$ and $W_t \leq \frac{\phi s_t - \psi}{\rho}$ from (A.4), and (A.5) now reads

$$\begin{aligned} \mathbf{E}\Pi_{it} - \phi + \phi \ln\left(\frac{\phi s_t}{\rho W_t + \psi}\right) &= \gamma + \phi \ln\left(\frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi}\right) \\ \mathbf{E}\Pi_{it} - (\phi + \gamma) &= \phi \ln\left(\frac{W_t \ell_t - \gamma}{\phi}\right) \end{aligned}$$

Note from (A.6) that the R.H.S. must be a negative number. On the other hand, $\mathbf{E}\Pi_{it} \geq \phi + \gamma$ implies that the L.H.S. is non-negative. Hence, we have a contradiction.

- The remaining task for the case of $n_{wt} = \frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi}$ is to make sure that $\mathbf{E}\Pi_{it} = W_t \ell_t$ when $\mathbf{E}\Pi_{it} < \phi + \gamma$. Re-writing (A.5) for $n_{wt} = \frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi}$ and $n_{it} = \frac{(\mathbf{E}\Pi_{it} - \gamma) s_t}{\rho W_t + \psi}$ confirms this:

$$\begin{aligned} \gamma + \phi \ln\left(\frac{(\mathbf{E}\Pi_{it} - \gamma) s_t}{\rho W_t + \psi}\right) &= \gamma + \phi \ln\left(\frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi}\right) \\ \ln\left(\frac{(\mathbf{E}\Pi_{it} - \gamma) s_t}{\rho W_t + \psi}\right) &= \ln\left(\frac{(W_t \ell_t - \gamma) s_t}{\rho W_t + \psi}\right) \\ \mathbf{E}\Pi_{it} &= W_t \ell_t \end{aligned}$$

- The Case of $n_{wt} = \frac{\phi s_t}{\rho W_t + \psi}$:

In this case, we must have from (A.1) that

$$W_t \in \left[\frac{\phi + \gamma}{\ell_t}, \frac{\phi s_t - \psi}{\rho}\right]$$

- $n_{it} = 1$ again leads to a contradiction because (A.4) implies $W_t > \frac{\phi s_t - \psi}{\rho}$.

– $n_{it} = \frac{(\mathbf{E}\Pi_{it} - \gamma)s_t}{\rho W_t + \psi}$ requires $\mathbf{E}\Pi_{it} < \phi + \gamma$. (A.5) in this case reads

$$\begin{aligned} \gamma + \phi \ln\left(\frac{(\mathbf{E}\Pi_{it} - \gamma)s_t}{\rho W_t + \psi}\right) &= W_t \ell_t - (\phi) + \phi \ln\left(\frac{\phi s_t}{\rho W_t + \psi}\right) \\ \phi \ln\left(\frac{\mathbf{E}\Pi_{it} - \gamma}{\phi}\right) &= W_t \ell_t - (\phi + \gamma) \end{aligned}$$

Once again, we have a contradiction because the L.H.S. is negative and the R.H.S. is non-negative.

– Note that $n_{it} = \frac{\phi s_t}{\rho W_t + \psi}$ as in the previous case confirms $\mathbf{E}\Pi_{it} = W_t \ell_t$ via (A.5).

- The Case of $n_{wt} = 1$:

In this case, we must have from (A.1) that

$$W_t > \frac{\phi s_t - \psi}{\rho} \tag{A.7}$$

– $n_{it} = \frac{\phi s_t}{\rho W_t + \psi}$ leads to a contradiction because (A.4) implies $W_t \leq \frac{\phi s_t - \psi}{\rho}$.

– $n_{it} = \frac{(\mathbf{E}\Pi_{it} - \gamma)s_t}{\rho W_t + \psi}$ requires $\mathbf{E}\Pi_{it} < \phi + \gamma$. (A.5) in this case reads

$$\begin{aligned} \gamma + \phi \ln\left(\frac{(\mathbf{E}\Pi_{it} - \gamma)s_t}{\rho W_t + \psi}\right) &= W_t \ell_t - (\psi + \rho W_t)\left(\frac{1}{s_t}\right) + \phi \ln(1) \\ \phi \ln\left(\frac{\mathbf{E}\Pi_{it} - \gamma}{\frac{\rho W_t + \psi}{s_t}}\right) &= W_t \ell_t - \gamma - \left(\frac{\psi + \rho W_t}{s_t}\right) \\ &= W_t\left(\frac{\ell_t s_t - \rho}{s_t}\right) - \left(\frac{\gamma s_t + \psi}{s_t}\right) \\ &= \left(\frac{\ell_t s_t - \rho}{s_t}\right)\left(W_t - \frac{\gamma s_t + \psi}{\ell_t s_t - \rho}\right) \end{aligned}$$

Since (A.7) implies $\frac{\rho W_t + \psi}{s_t} > \phi$, the L.H.S. of this last equation is negative given $\mathbf{E}\Pi_{it} < \phi + \gamma$. The R.H.S. is on the other hand positive because

$$W_t > \frac{\phi s_t - \psi}{\rho} > \frac{\phi + \gamma}{\ell_t}$$

where the second inequality implies

$$\phi > \frac{\psi \ell_t + \rho\gamma}{\ell_t s_t - \rho}$$

Since we have $W_t \ell_t - \gamma > \phi$ as well, $W_t \ell_t - \gamma > \frac{\psi \ell_t + \rho\gamma}{\ell_t s_t - \rho}$ implies that $W_t > \frac{\gamma s_t + \psi}{\ell_t s_t - \rho}$.

– Finally, note that $n_{it} = 1$ as in the previous cases confirms $\mathbf{E}\Pi_{it} = W_t \ell_t$ via (A.5).

Hence, we conclude that the equal utilities restriction $\mathbf{E}U_{it} = U_{wt}$ is satisfied with $\mathbf{E}\Pi_{it} = W_t \ell_t$ and $n_{wt} = n_{it} = n_t$. $\square$

To proceed, notice that $\mathbf{E}\Pi_{it} = W_t \ell_t$ and (A.3) solve W_t given the unique solution $a_t \geq 0$ that follows from (A.2). Next, given a_t, (3.9) implies $h_{rit} = h_{rt}$, and (3.18) then returns $h_{mit} = h_{mt}$ for the given level of ℓ_t. Given W_t, on the other hand, the unique SGE levels of $\left(H_{Tt}, N_{Tt}, Y_{Tt}\right)$ follow respectively from (3.20), (3.21) and (3.12′). Note that the realizations of stochastic variables z_{it} and X_{it} simply follow from (3.8) and (3.7), respectively. Given productivity X_{it}, (3.15), (3.16) and (3.5) respectively solve the unique SGE levels of $\left(h_{wit}, \Pi_{it}, Y_{it}\right)$. (3.22) solves C_{wt}, and (3.25) solves C_{it}. Thus, only E_t and Y_t remain to be solved.

What solves E_t is the labor market clearing condition (3.30). To see this, first recall that the arrival rate a_t is common across entrepreneurs. This and the fact that invention events are independent across entrepreneurs imply, via (Borel's version of) the law of large numbers, that the ex post fraction of entrepreneurs with $z \geq 0$ inventions for any given a_t is equal to the ex ante Poisson probability $\frac{a_t^z \exp\left(-a_t\right)}{z!}$ of achieving $z \geq 0$ inventions.

This property allows us to write

$$\int_{0}^{E_t} h_{wit} \mathrm{d}i = E_t \sum_{z=0}^{\infty} \left[\frac{a_t^z \exp\left(-a_t\right)}{z!} \right] h_{wt}(z) \tag{A.8}$$

where $h_{wt}(z)$ reads, as implied by (3.15),

$$h_{wt}(z) \equiv \lambda^{\frac{1}{1-\lambda}} \sigma^{\left(\frac{\lambda}{1-\lambda}\right)z} \left(\frac{\overline{X}_t^{\frac{\lambda}{1-\lambda}}}{W_t^{\frac{1}{1-\lambda}}} \right) \left(\ell_t - \frac{a_t}{\theta \xi \left(K_t\right)} \right)$$

Applying the reasoning of Lemma 1 to the R.H.S. of (A.8) implies

$$\int_{0}^{E_t} h_{wit} \mathrm{d}i = E_t \exp\left(\Sigma a_t\right) \lambda^{\frac{1}{1-\lambda}} \left(\frac{\overline{X}_t^{\frac{\lambda}{1-\lambda}}}{W_t^{\frac{1}{1-\lambda}}} \right) \left(\ell_t - \frac{a_t}{\theta \xi \left(K_t\right)} \right)$$

This last equation and (3.30) then solve E_t.

Finally, given E_t, Y_t is solved from (3.19) using the above reasoning

$$\int_{0}^{E_t} Y_{it} \mathrm{d}i = E_t \sum_{z=0}^{\infty} \left[\frac{a_t^z \exp\left(-a_t\right)}{z!} \right] Y_t(z)$$

where $Y_t(z)$ satisfies $Y_t(z) \equiv \lambda^{\frac{\lambda}{1-\lambda}} \sigma^{\left(\frac{\lambda}{1-\lambda}\right)z} \left(\overline{X}_t^{\frac{\lambda}{1-\lambda}} / W_t^{\frac{1}{1-\lambda}} \right) \left(\ell_t - a_t / \theta \xi \left(K_t\right) \right)$. ■

Proof of Proposition 2:

The invention threshold simply follows from (A.2). The (symmetric) solution of a_t is at boundary, i.e. $a_t = 0$, if

$$\Lambda \left(\frac{\overline{X}_t}{W_t} \right)^{\Gamma} \left[-\frac{1}{\theta \xi \left(K_t\right)} + \Sigma \ell_t \right] < 0$$

Since $\Lambda > 0$ and $\left(\overline{X}_t / W_t\right)^{\Gamma} > 0$, this inequality implies

$$\xi\left(K_t\right)\ell_t < (\theta\Sigma)^{-1} = \left[\theta\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)\right]^{-1}$$

The interior solution $a_t = \theta\xi\left(K_t\right)\ell_t - \left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1} > 0$ follows, again, from (A.2). ■

Proof of Proposition 3:

As the proof of Lemma A.1 shows, the expected profit $\mathbf{E}\Pi_{it}$ in the unique SGE is equal to $W_t \ell_t$. Hence, (A.1) and (A.4) imply the desired result given $n_{wt} = n_{it} = n_t$. Furthermore, since s_t is common across entrepreneurs and workers, we have $b_{wt} = b_{it} = b_t = n_t / s_t$. ■

Proof of Proposition 4:

The existence and the uniqueness of period-t SGE and that the laws of motion for endogenous state variables, i.e. (3.1), (3.13), (4.11) and (4.12), are all one-to-one functions imply the existence and the uniqueness of the DGE for the entire history from $t = 0$ to $t \to \infty$. ■

Proof of Proposition 5:

First note that $\ell_t E_t > 0$ for all t implies $K_{t+1} - K_t > 0$ for all t. Next, it is assumed that $K_t^{tr} < \infty$. Thus the continuing growth of the stock K_t of useful discoveries eventually makes the inventive activity optimal at some period t^{tr} where $K_{t^{tr}} \geq K_{t^{tr}}^{tr}$. ■

Appendix B

The Conditional Dynamical System

As noted earlier, the dynamical system of the model cannot be rewritten as a simple autonomous system of normalized variables due to its severe non-linearity. A formal discussion of global stability however is still feasible since the closed-form solution of the model's unique SGE allows us to study the qualitative properties of a conditional dynamical system with a phase diagram. The modest purpose of this appendix is to show why the asymptotic equilibrium is globally stable.

The conditional dynamical system to be considered, possibly the simplest among all, is that of $(\overline{X}_t, q_t)$ where we define q_t as follows:

$$q_t \equiv \frac{N_t}{K_t}$$

The asymptotic equilibrium on $(\overline{X}_t, q_t)$ plane is now characterized by $\overline{X}_t \to \infty$ and $q_t \to 0$ for $t \to \infty$.

This system is conditional on endogenous state variables $(N_t, K_t, \overline{X}_t, X_{Tt})$ and exogenous state variables (s_t, ℓ_t). For notational ease, define $\mathbf{Z}_t \equiv (N_t, K_t, X_{Tt}, s_t, \ell_t)$ as the vector of state variables excluding $\overline{X}_t$. In what follows, the notation is slightly abused by redefining some of the model variables as functions of $\mathbf{Z}_t$ even if some elements of $\mathbf{Z}_t$ do not affect the variable in the question.

The first equation of the $(\overline{X}_t, q_t)$ system governs $\overline{X}_t$ and rewritten here as

$$\frac{\overline{X}_{t+1}}{\overline{X}_t} = \exp\left[(\sigma - 1)a(\mathbf{Z}_t)\right]$$

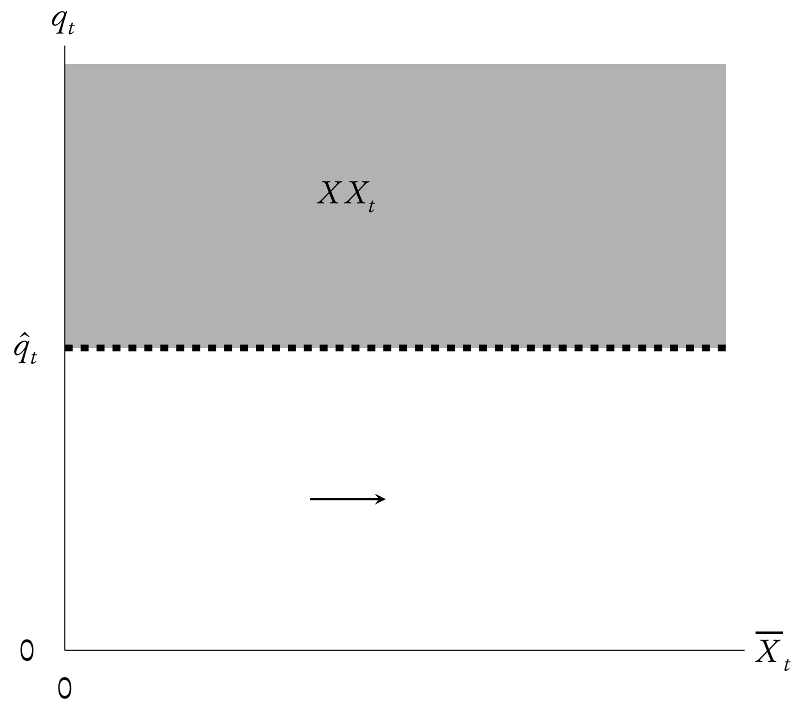

Figure B.1: The XX_t Set and the Dynamics of $\overline{X}_t$

where the arrival rate $a(\mathbf{Z}_t)$ is as in (4.1). On $(\overline{X}_t, q_t)$ plane, the set of vectors satisfying $\overline{X}_{t+1} = \overline{X}_t$ is clearly

$$q_t \geq \hat{q}_t \equiv \hat{q}(\mathbf{Z}_t) \equiv \frac{N_t}{K_t^{tr}} = \frac{N_t}{\xi^{-1}\left[\theta^{-1}\left(\sigma^{\frac{\lambda}{1-\lambda}} - 1\right)^{-1} \ell_t^{-1}\right]} \qquad (XX_t)$$

and $\overline{X}_t$ grows wherever $q_t < \hat{q}_t$, i.e. wherever, for any given value of adult population N_t, the stock of discoveries is sufficiently large ($K_t > K_t^{tr}$). Importantly, for any given value of $\overline{X}_t$, $\hat{q}_t$ is increasing in N_t and ℓ_t. Figure B.1 pictures the dynamics of $\overline{X}_t$ regardless of how q_t changes in time.

The second equation of the system, the one that governs q_t, reads

$$\frac{q_{t+1}}{q_t} \equiv \frac{N_{t+1}/N_t}{K_{t+1}/K_t} = \frac{n(\mathbf{Z}_t, \overline{X}_t)}{1 + \omega \ell_t e(\mathbf{Z}_t, \overline{X}_t) q_t} = \frac{n(\mathbf{Z}_t, \overline{X}_t)}{1 + \omega \ell_t \left(f^{NM}(\mathbf{Z}_t, \overline{X}_t) - \frac{\rho n(\mathbf{Z}_t, \overline{X}_t)}{s_t \ell_t}\right) q_t}$$

where $n(\mathbf{Z}_t, \overline{X}_t) = n_t$ is the level of net fertility, $e(\mathbf{Z}_t, \overline{X}_t) = E_t/N_t$ is the share of entrepreneurs in adult population, and $f^{NM}(\mathbf{Z}_t,\overline{X}_t) = 1-(N_{Tt}/N_t)$ is the labor share of the modern sector. The locus of vectors satisfying $q_{t+1} = q_t$ on $(\overline{X}_t, q_t)$ plane is thus

$$q_t = \frac{n(\mathbf{Z}_t,\overline{X}_t)-1}{\omega\ell_t\left(f^{NM}(\mathbf{Z}_t,\overline{X}_t)-\frac{\rho n(\mathbf{Z}_t,\overline{X}_t)}{s_t\ell_t}\right)} \tag{qq_t}$$

As usual, the shape of qq_t locus on $(\overline{X}_t, q_t)$ plane and the dynamics of q_t below and above this locus are of interest. Regarding the latter, we simply have $\partial(q_{t+1}/q_t)/\partial q_t < 0$ for all q_t. Thus, q_t increases wherever q_t is below the qq_t locus and it decreases otherwise.

The shape of the qq_t locus is determined by $f^{NM}(\mathbf{Z}_t,\overline{X}_t)$ and $n(\mathbf{Z}_t,\overline{X}_t)$. Regarding the former, we know that $f^{NM}(\mathbf{Z}_t,\overline{X}_t)$ is a strictly increasing function of $\overline{X}_t$ for any $(\mathbf{Z}_t,\overline{X}_t)$ due to the decline of the traditional sector with growing $\overline{X}_t$. Net fertility $n(\mathbf{Z}_t,\overline{X}_t)$, on the other hand, changes with $\overline{X}_t$ non-monotonically since there exist three regimes of net fertility determined by $\overline{X}_t$ given $\mathbf{Z}_t$. These regimes are separated by two thresholds of $\overline{X}_t$ such that

$$\begin{aligned}
\frac{\partial n(\mathbf{Z}_t,\overline{X}_t)}{\partial \overline{X}_t} > 0 \quad & \text{if } \overline{X}_t < \hat{\overline{X}}_t^{I} \equiv \left[\frac{\phi+\gamma}{\ell_t(1-\lambda)^{1-\lambda}\lambda^{\lambda}\delta\left(a_t,K_t,\ell_t\right)}\right]^{\frac{1}{\lambda}} \\
n(\mathbf{Z}_t,\overline{X}_t) = 1 \quad & \text{if } \overline{X}_t > \hat{\overline{X}}_t^{II} \equiv \left[\frac{\phi s_t-\psi}{\rho(1-\lambda)^{1-\lambda}\lambda^{\lambda}\delta\left(a_t,K_t,\ell_t\right)}\right]^{\frac{1}{\lambda}} \\
\frac{\partial n(\mathbf{Z}_t,\overline{X}_t)}{\partial \overline{X}_t} < 0 \quad & \text{if } \overline{X}_t \in \left[\hat{\overline{X}}_t^{I}, \hat{\overline{X}}_t^{II}\right]
\end{aligned}$$

For $\overline{X}_t > \hat{\overline{X}}_t^{II}$, the qq_t locus is horizontal at $q_t = 0$ since net fertility n_t is equal to unity. For $\overline{X}_t \in \left[\hat{\overline{X}}_t^{I}, \hat{\overline{X}}_t^{II}\right]$, the qq_t locus is downward-sloping with decreasing $n(\mathbf{Z}_t,\overline{X}_t)$ and increasing $f^{NM}(\mathbf{Z}_t,\overline{X}_t)$ with respect to $\overline{X}_t$. For $\overline{X}_t < \hat{\overline{X}}_t^{I}$, however, the shape of the qq_t locus remains ambiguous because $n(\mathbf{Z}_t,\overline{X}_t)$ is increasing in $\overline{X}_t$ in this case. The sign of

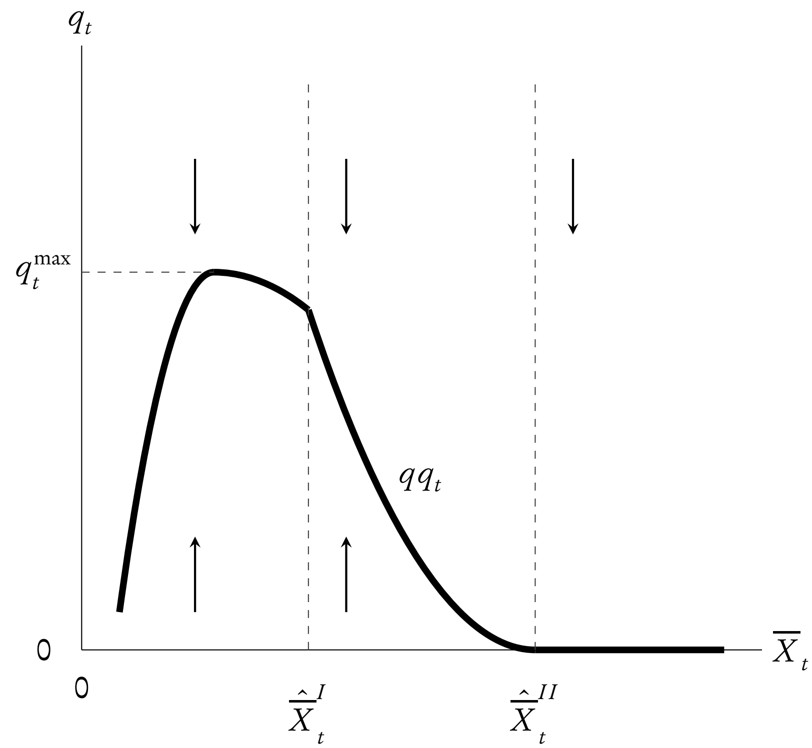

Figure B.2: The qq_t Locus and the Dynamics of q_t

the slope of the qq_t locus is equal to the sign of

$$n_X(\mathbf{Z}_t,\overline{X}_t)\left(f^{NM}(\mathbf{Z}_t,\overline{X}_t)-\rho/s_t\ell_t\right)-f_X^{NM}(\mathbf{Z}_t,\overline{X}_t)\left(n(\mathbf{Z}_t,\overline{X}_t)-1\right)\gtreqless 0$$

where $n_X(\bullet,\bullet)$ and $f_X^{NM}(\bullet,\bullet)$ denote associated partial derivatives with respect to $\overline{X}_t$. An inspection of second derivatives with respect to $\overline{X}_t$ further indicates that there may exist local maxima and local minima of q_t for $0<\overline{X}_t<\hat{\overline{X}}_t^I$.

The ambiguity, fortunately, does not affect the global stability result. The reason, as it shall become clear below, is that, for any $0<\overline{X}_t<\hat{\overline{X}}_t^I$, we have

$$n(\mathbf{Z}_t,\overline{X}_t)\in\left[1,\frac{\phi\ell_t s_t}{\rho(\phi+\gamma)+\psi\ell_t}\right]\quad\text{and}\quad f^{NM}(\mathbf{Z}_t,\overline{X}_t)\in(0,1]$$

These imply, together with the uniqueness of q_t given $(\mathbf{Z}_t,\overline{X}_t)$, that there exists a unique global maximum $q_t^{\max}<\infty$ of the qq_t locus for $0<\overline{X}_t<\hat{\overline{X}}_t^I$ regardless of its local

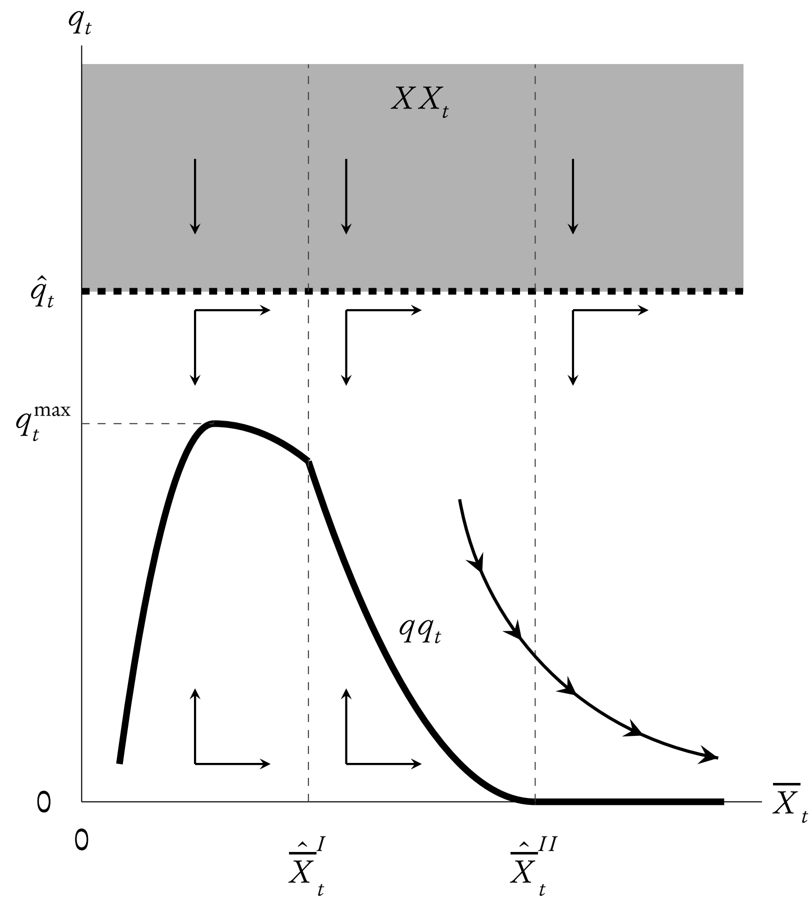

Figure B.3: The Convergence to the Asymptotic Equilibrium

maxima and local minima.

Figure B.2 pictures one possible characterization of the qq_t locus and the associated dynamics of q_t. With the confidence following from the existence of unique $q_t^{\max}$ which is bounded above, the rest of the analysis is carried out with this possibility.

In Figure B.3, the XX_t set and the qq_t locus are drawn together. Note that the following are subject to change in time due to the dependence on $\mathbf{Z}_t$:

- the XX_t set,
- the $\hat{q}_t$ threshold,
- the qq_t locus,
- the $q_t^{\max}$ maximum,

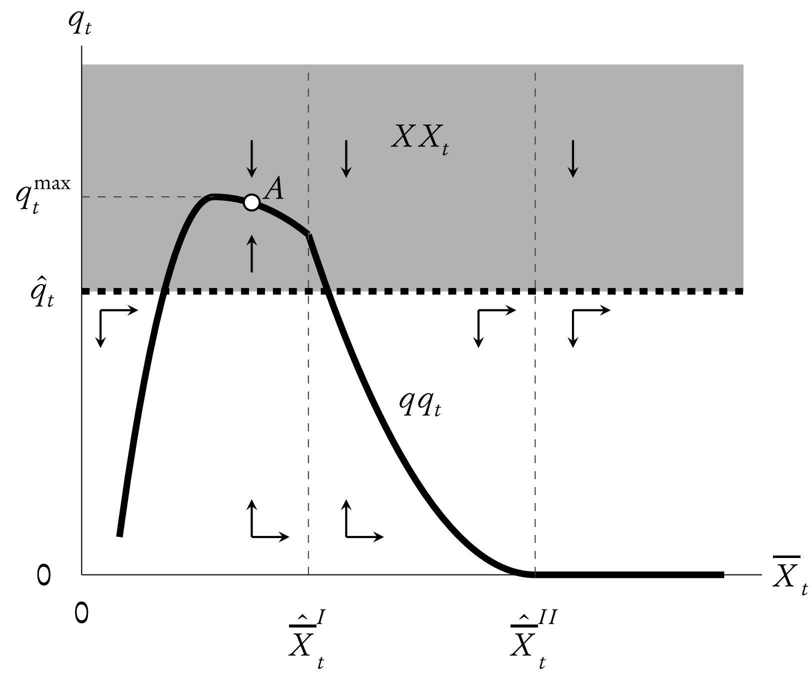

Figure B.4: A Quasi-Static Equilibrium without Productivity Growth

- the $\hat{\overline{X}}_t^{I}$ and $\hat{\overline{X}}_t^{II}$ thresholds.

The claim here is that, even though these elements of the conditional dynamical system change in not easily predictable ways because of evolving $\mathbf{Z}_t$, $\hat{q}_t$ becomes strictly greater than $q_t^{\max}$, once and for all, at some finite t. To see why, notice that net fertility n_t is greater than unity for any $\overline{X}_t$ that is less than $\hat{\overline{X}}_t^{II}$. This in turn implies that $\hat{q}_t$ grows, even for constant ℓ_t, as long as $\overline{X}_t$ is sufficiently low. Put differently, the set XX_t continuously moves upwards on $(\overline{X}_t, q_t)$ plane along the trajectory towards the industrial revolution.[38] Once the system is characterized by $\hat{q}_t > q_t^{\max}$, it always moves towards the asymptotic equilibrium of $\overline{X}_t \to \infty$ and $q_t \to 0$ because the qualitative properties of the system remains unchanged. When $\overline{X}_t$ passes its second threshold, $n_t = n^\star = 1$ lets $\hat{q}_t$ stabilize through $N_t = N^\star$ and ℓ_t converging to unity.

In Figure B.4, a quasi-static equilibrium without productivity growth in the modern sector is pictured to provide more insight on the dynamic properties of the model.

38. Needless to say, assuming that the industrial revolution is possible implies $\hat{q}_t > 0$ for all t.

Recalling the earlier characterization of the initial stagnation equilibrium, if it is again assumed that s_t and ℓ_t are constant and that the exogenous productivity growth in the traditional sector just matches the effect of slowly increasing population, the point A in the figure represents a quasi-static equilibrium of the model. In this equilibrium, a knife-edge relation is *endogenously* established between population growth and the growth of the stock of useful discoveries, making $q_{t+1} = q_t$. Since the labor share of the traditional sector does not change in this equilibrium, the quasi-statis can be prolonged in calendar time until the stock of useful discoveries, or, equivalently, the level of population given constant q_t, is sufficiently large to imply $\hat{q}_t > q_t^{\max}$.

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